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2-Absorbing R-ideals of modules over near rings

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Abstract

It is known that rings and modules over rings are related algebraic structures. Moreover, near rings and modules over near rings are generalized algebraic structures of rings and modules over rings, respectively. In this paper, we introduce and study 2-absorbing ideals of near rings and 2-absorbing R-ideals of modules over near rings which are extended from prime ideals of near rings and prime R-ideals of modules over near rings, respectively.

Keywords: 2-absorbing ideals, 2-absorbing R-ideals, near rings, modules over near rings.

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1 Introduction

In 2007, Badawi [1] introduced the concept of 2-absorbing ideals of commutative rings with identity, which is a generalization of prime ideals, and investigated some properties. He defined a **2-absorbing ideal** P of a commutative ring R with identity to be a proper ideal of R and if whenever $a, b, c \in R$ with $abc \in P$, then $ab \in P$ or $bc \in P$ or $ac \in P$. In 2011, Darani and Soheilnia [4] introduced the concept of 2-absorbing submodules of modules over commutative rings with identities. A proper submodule P of a module M over a commutative ring R with identity is said to be a **2-absorbing submodule** of M if whenever $a, b \in R$ and $m \in M$ with $abm \in P$, then $abM \subseteq P$ or $am \in P$ or $bm \in P$. One can see that 2-absorbing submodules are generalization of prime submodules. Moreover, it is obvious that 2-absorbing ideals are special cases of 2-absorbing submodules.

It is known that a near ring is an algebraic structure similar to a ring. In 1991, Groenewald [7] introduced the notion of prime ideals of near rings. Moreover, Booth and Groenewald [2] extended prime ideals of near rings to prime R-ideals of modules over near rings.

In this paper, we aim to study the notion that generalizes prime R-ideals of modules over near rings in the same way as prime submodules of modules over rings were extended, called 2-absorbing R-ideals. Furthermore, we investigate some properties of 2-absorbing R-ideals of decomposable modules over near rings.

2 Preliminaries

We collect definitions of near rings and modules over near rings as well as present some results which are used in this paper.

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2.1 Near Rings

In 1905, Dickson [6] showed that there exists a near field which is an algebraic structure similar to a field except that the multiplication is not necessarily commutative and at least one distributive law holds. Some years later, the concept of near rings were introduced. A near ring is a generalization of a ring whose two axioms are omitted, namely, the addition is not necessarily abelian and the multiplication distributes over the addition is applied on a left or a right side.

Definition 2.1. [9] A set R together with two operations of addition and multiplication is called a **near ring** if the following conditions are satisfied:

(i) $(R,+)$ is a group where the additive identity of $(R,+)$ is denoted by 0,

(ii) $(R,\cdot)$ is a semigroup, and

(iii) $(a+b)\cdot c = a\cdot c + b\cdot c$ for all $a,b,c\in R$.

For any $a,b\in R$, we may write ab instead of $a\cdot b$.

Definition 2.2. [9] A near ring R is called a **near ring with identity** if there is an element $b\in R$ such that $ab = a = ba$ for all $a\in R$; we say that b is the **(multiplicative) identity** of the near ring R.

If R is a near ring, then it is always true that $0r = 0$ for all $r\in R$ because $0r = (r-r)r = rr - rr = 0$. However, the following example shows that $r0$ is not necessarily equal to 0.

Example 2.3. Let $R = \{0,1\}$ be the set with addition and multiplication given by the following tables:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	1	1

Then $(R,+,\cdot)$ is a near ring without identity which is not a ring because $1(1+0)\neq 1\cdot 1 + 1\cdot 0$.

Definition 2.4. [9] A near ring R is called a **zero symmetric near ring** if $r0 = 0$ for all $r\in R$.

The near ring given in Example 2.3 is not a zero symmetric near ring because $1\cdot 0 = 1 \neq 0$.

Example 2.5. Let $R = \{0,1,a,b\}$ be the set with addition and multiplication given by the following tables:

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	0	0

Then $(R,+,\cdot)$ is a zero symmetric near ring with identity 1.

Definition 2.6. [9] A subset H of a near ring R is called an **R-subgroup** of R if

(i) $(H,+)$ is a subgroup of $(R,+)$,

(ii) $HR\subseteq H$ where $HR = \{hr : h\in H \text{ and } r\in R\}$, and

(iii) $RH\subseteq H$ where $RH = \{rh : h\in H \text{ and } r\in R\}$.

Moreover, if the conditions (i) and (ii) are satisfied, then H is called a **right R-subgroup**. If the conditions (i) and (iii) are satisfied, then H is called a **left R-subgroup**.

Definition 2.7. [9] A subset I of a near ring R is called an **ideal** of R if

(i) $(I, +)$ is a normal subgroup of $(R, +)$,

(ii) $IR \subseteq I$, and

(iii) $r_1 (r_2 + k) - r_1 r_2 \in I$ for all $r_1, r_2 \in R$ and $k \in I$.

Nevertheless, if I satisfies the conditions (i) and (ii), then I is called a **right ideal** of R, while I is called a **left ideal** of R if the conditions (i) and (iii) are satisfied.

In general, R-subgroups and ideals of near rings may not imply one another. However, if $(R, +)$ is an abelian group, then left R-subgroups and left ideals of R are identical. Although R is a near ring such that $(R, +)$ is abelian, right R-subgroups are not necessarily right ideals and vice versa. This is because near rings may have only one distributive law. However, if R is a zero symmetric near ring, then every ideal of R is an R-subgroup of R. In fact, if I is an ideal of a zero symmetric near ring R, then $RI \subseteq I$ because $rk = r(0 + k) - r0 \in I$ for all $r \in R$ and $k \in I$.

Definition 2.8. [9] A proper ideal P of a near ring R is called a **prime ideal** of R if for all $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$.

Example 2.9. Consider the set $R = \{0, x, y, z\}$ with addition and multiplication given by the following tables:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	x	y	z
y	0	0	0	0
z	0	x	y	z

Then R is a near ring, see [8]. Moreover, one can notice that all ideals of R are $\{0\}, \{0, x\}, \{0, y\}$ and $\{0, z\}$. In addition, $\{0, y\}$ is the only prime ideal of R. The ideals $\{0\}, \{0, x\}$ and $\{0, z\}$ are not prime ideals because $yRy = \{0\}$ which is a subset of $\{0\}, \{0, x\}$ and $\{0, z\}$ but $y \notin \{0\}, y \notin \{0, x\}$ and $y \notin \{0, z\}$.

2.2 Modules over Near Rings

We know that rings are special cases of modules over rings. It is natural to introduce the notion of modules over near rings which are generalization of near rings. It turns out that modules over near rings also are generalization of modules over rings.

Definition 2.10. [9] Let R be a near ring and $(M, +)$ a group. Then M is called a **module over a near ring** R (or an R**-module**) if there exists a **scalar multiplication** $\cdot : R \times M \to M$ such that for all $r_1, r_2 \in R$ and $m \in M$,

(i) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$, and

(ii) $(r_1 r_2) \cdot m = r_1 (r_2 \cdot m)$.

For any $r \in R$ and $m \in M$, we may write rm instead of $r \cdot m$. It is obvious that every near ring is a module over itself.

Example 2.11. Let $(R = \{0, 1\}, +, \cdot)$ be the near ring which is not a ring given in Example 2.3 and $M = \{0, a\}$ be the set with addition $+$ on M and scalar multiplication $\odot : R \times M \to M$ given by the following tables:

$+$	0	a
0	0	a
a	a	0

$\odot$	0	a
0	0	a
1	0	a

Then M is a module over the near ring R.

Definition 2.12. [9] Let R be a near ring. A subgroup N of an R-module M is called an **R-submodule** of M if $rn \in N$ for all $r \in R$ and $n \in N$.

Definition 2.13. [9] Let R be a near ring. A normal subgroup N of an R-module M is called an **R-ideal** of M if $r(m+n) - rm \in N$ for all $r \in R$, $m \in M$ and $n \in N$.

Let R be a near ring. Consider $M = R$ as an R-module. Then R-ideals of M are the same as left ideals of R and R-submodules of M are left R-subgroups of R. The following examples show that R-submodules and R-ideals do not imply each other.

Example 2.14. Let R be the near ring given in Example 2.3. We consider $M = R$ as an R-module. Then $\{0\}$ is an R-ideal of M but $\{0\}$ is not an R-submodule of M because $1 \cdot 0 = 1 \notin \{0\}$.

Example 2.15. Let R be the near ring given in Example 2.5. Let $M = R$ be an R-module. Then all R-submodules of M are $\{0\}, \{0,a\}, \{0,b\}$ and M. Moreover, all R-ideals of M are $\{0\}, \{0,b\}$ and M. Note that $\{0,a\}$ is not an R-ideal because $a(b+a) - ab = a(1) - b = a + b = 1 \notin \{0,a\}$. Thus $\{0,a\}$ is an R-submodule but not an R-ideal of M.

The next proposition provides the condition that makes each R-ideal be an R-submodule.

Proposition 2.16. *If R is a zero symmetric near ring, then every R-ideal of an R-module M is an R-submodule of M.*

Proof. Assume that R is a zero symmetric near ring. Let N be an R-ideal of an R-module M. Then N is a normal subgroup of M. Next, we show that $rn \in N$ for all $r \in R$ and $n \in N$. Let $r \in R$ and $n \in N$. Since R is a zero symmetric near ring, $r0 = 0$. And we have $rn = r(0+n) - r0 \in N$ because N is an R-ideal of M and $n \in N$. Therefore, N is an R-submodule of M. $\square$

The previous proposition shows that every R-ideal is an R-submodule when R is a zero symmetric near ring. However, R-submodules are not necessarily R-ideals even R is a zero symmetric near ring. For example, see Example 2.15.

It is known that the intersection of submodules of modules over rings is a submodule. Next, we consider the intersection of R-submodules as well as the intersection of R-ideals of modules over near rings.

Proposition 2.17. *Let N and K be R-submodules of an R-module M. Then $N \cap K$ is an R-submodule of M.*

Proof. The proof is straightforward. $\square$

Proposition 2.18. *Let N and K be R-ideals of an R-module M. Then $N \cap K$ is an R-ideal of M.*

Proof. Since N and K are R-ideals of M, we obtain that N and K are normal subgroups of M so that $N \cap K$ is a normal subgroup of M. Let $r \in R, m \in M$ and $n \in N \cap K$. Then $n \in N$ and $n \in K$. Since N and K are R-ideals of M, it follows that $r(m+n) - rm \in N$ and $r(m+n) - rm \in K$. That is $r(m+n) - rm \in N \cap K$. Therefore, $N \cap K$ is an R-ideal of M. $\square$

One can see from the definitions of R-submodules and R-ideals that R-ideals are normal subgroups of R-modules but R-submodules are not necessary. Consequently, these allow us to define quotient modules over near rings by using R-ideals.

Proposition 2.19. *Let N be an R-ideal of an R-module M. Set $M/N = \{m + N : m \in M\}$. Define the addition $+$ on M/N and the scalar multiplication $\cdot$ by*

$$(m+N) + (n+N) = (m+n) + N \quad \text{and} \quad r \cdot (m+N) = rm + N$$

for all $r \in R$ and $m, n \in M$. Then $(M/N, +, \cdot)$ is an R-module and called the ***quotient module over a near ring****.*

Proof. Since N is an R-ideal of M, it follows that $(N,+)$ is a normal subgroup of $(M,+)$. Thus, $(M/N,+)$ is a group. Now, we show that the scalar multiplication is well-defined. Let $x,y \in M$ and $r \in R$. Assume that $x+N = y+N$. Then $-y+x \in N$ and thus $rx - ry = r(y+(-y+x)) - ry \in N$ because N is an R-ideal of M. Since N is a normal subgroup of M and $rx - ry \in N$, it follows that $-ry+rx = -ry+(rx-ry)+ry \in N$. Hence the scalar multiplication is well-defined. Next, we show that $(r_1+r_2)(x+N) = r_1(x+N) + r_2(x+N)$ and $(r_1r_2)(x+N) = r_1(r_2(x+N))$ for all $r_1, r_2 \in R$. Let $r_1, r_2 \in R$. First, $(r_1+r_2)(x+N) = (r_1+r_2)x+N = (r_1x+r_2x)+N = (r_1x+N)+(r_2x+N) = r_1(x+N)+r_2(x+N)$. Finally, $(r_1r_2)(x+N) = (r_1r_2)x+N = r_1(r_2x)+N = r_1(r_2x+N) = r_1(r_2(x+N))$. Therefore, $(M/N,+,\cdot)$ is an R-module. □

2.3 2-Absorbing R-Ideals

Prime submodules of modules over commutative rings with identities were introduced by Dauns [5]. He defined a **prime submodule** N of a module M over a commutative ring R with identity to be a proper submodule N of M and if $rm \in N$ implies $rM \subseteq N$ or $m \in N$ for all $r \in R$ and $m \in M$. Recently, prime submodules of modules over rings were developed to 2-absorbing submodules of modules over rings, see [4]. Moreover, Booth and Groenewald extended, in [2], prime ideals of near rings to prime R-ideals of modules over near rings. In this part, we extend the idea of prime ideals of near rings and prime R-ideals of modules over near rings to 2-absorbing ideals of near rings and 2-absorbing R-ideals of modules over near rings, respectively. In addition, some basic results of these are provided at the end.

Definition 2.20. [2] Let R be a near ring and N be a proper R-ideal of an R-module M. Then N is called a **prime R-ideal** of M if $rRm \subseteq N$ implies $rM \subseteq N$ or $m \in N$ for all $r \in R$ and $m \in M$.

Example 2.21. Recall from Example 2.14 that $\{0\}$ is the only proper R-ideal of M. And it is easy to check that $\{0\}$ is the only prime R-ideal of M.

Example 2.22. Recall from Example 2.15 that $\{0\}$ and $\{0,b\}$ are the only proper R-ideals of M. One can check that $\{0,b\}$ is the only prime R-ideal of M. Note that $\{0\}$ is not a prime R-ideal of M because $b \cdot b = 0$ but $b \notin \{0\}$.

Next, we give the definitions of 2-absorbing ideals of near rings and 2-absorbing R-ideals of modules over near rings.

Definition 2.23. Let P be a proper ideal of a near ring R. Then P is called a **2-absorbing ideal** of R if $aRbRc \subseteq P$ implies $ab \in P$ or $bc \in P$ or $ac \in P$ for all $a,b,c \in R$.

Definition 2.24. Let R be a near ring and N be a proper R-ideal of an R-module M. Then N is called a **2-absorbing R-ideal** of M if $aRbRm \subseteq N$ implies $abM \subseteq N$ or $am \in N$ or $bm \in N$ for all $a,b \in R$ and $m \in M$.

Badawi introduced, in [1], 2-absorbing ideals of rings and showed that every prime ideal of a ring is a 2-absorbing ideal. Later, Darani and Soheilnia provided the notion of 2-absorbing submodules of modules over rings and proved that every prime submodule of a module over a ring is a 2-absorbing submodule, see [4]. Consequently, we expect to obtain the similar result in term of "2-absorbing". Anyhow, the following result is needed.

Proposition 2.25. *Let R be a near ring and N be a prime R-ideal of an R-module M. If $aRbRm \subseteq N$ and $am \notin N$, then $bM \subseteq N$ for all $a,b \in R$ and $m \in M$.*

Proof. Let $a,b \in R$ and $m \in M$. Assume that $aRbRm \subseteq N$ and $am \notin N$. First, we show that $bRm \subseteq N$. Let $r \in R$. Then $aR(brm) \subseteq aR(bRm) \subseteq N$. Since N is a prime R-ideal, $aM \subseteq N$ or $brm \in N$. Then $brm \in N$ because $am \notin N$. That is $bRm \subseteq N$. Since N is a prime R-ideal and $am \notin N$, it follows that $m \notin N$ so that $bM \subseteq N$. □

Proposition 2.26. *If N is a prime R-ideal of an R-module M, then N is a 2-absorbing R-ideal of M.*

Proof. Assume that N is a prime R-ideal of an R-module M. Let $a, b \in R$ and $m \in M$. Assume that $aRbRm \subseteq N$ but $am \notin N$. Thus $bM \subseteq N$ by Proposition 2.25. Then $bm \in N$ and $abM \subseteq N$. Hence N is a 2-absorbing R-ideal of M. □

Proposition 2.26 guarantees that every prime R-ideal is a 2-absorbing R-ideal. But the converse does not necessarily hold. Example 2.22 provides that $\{0\}$ is not a prime R-ideal of M. However, $\{0\}$ is a 2-absorbing R-ideal of M. To see this, let $x, y, z \in R$. Assume that $xRyRz = \{0\}$. If $x = 0$ or $y = 0$ or $z = 0$, then $xy = 0$ or $xz = 0$ or $yz = 0$ because R is a zero symmetric near ring. Next, Suppose that each of x, y and z is not zero. There are 2 cases to be considered:

(i) at least two of x, y and z are 1, and

(ii) at most one of x, y and z are 1.

First, we consider Case(i). Without loss of generality, it suffices to assume that x and y are 1. It follows that $1R1Rz \neq \{0\}$ which is a contradiction. Thus Case(i) does not occur. Next, Case(ii) is considered. There are 3 possible choices of $xRyRz$, namely, $bRyRz$, $xRbRz$, or $xRyRb$. We obtain from the multiplication table in Example 2.5 that $xRyRb \neq \{0\}$ which is absurd. If $\{0\} = xRyRz = bRyRz$, then $by = 0$. Or, if $\{0\} = xRyRz = xRbRz$, then $bz = 0$. This shows that whenever $xRyRz = \{0\}$, then $xy = 0$ or $xz = 0$ or $yz = 0$. Therefore, $\{0\}$ is a 2-absorbing R-ideal of M.

3 Main Results

In this section, some properties of prime R-ideals and 2-absorbing R-ideals are presented. The first part is regarded intersections of prime R-ideals as well as relationships between prime (2-absorbing) R-ideals of an R-module and prime (2-absorbing) R-ideals of its quotient module. The other part is considered results on decomposable near rings.

In 2011, Darani and Soheilnia showed in [4] that the intersection of each pair of prime submodules of modules over rings is a 2-absorbing submodule. It is reasonable to extend this result to the intersection of each pair of prime R-ideals of modules over near rings.

Theorem 3.1. *The intersection of each pair of prime R-ideals of an R-module M is a 2-absorbing R-ideal of M.*

Proof. Let N and K be two prime R-ideals of M. If $N = K$, then $N \cap K$ is a prime R-ideal of M so that $N \cap K$ is a 2-absorbing R-ideal of M. Assume that N and K are distinct. Since N and K are proper R-ideals of M, it follows that $N \cap K$ is a proper R-ideal of M. Next, let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N \cap K$ but $am \notin N \cap K$ and $abM \nsubseteq N \cap K$. Then, we can conclude that (a) $am \notin N$ or $am \notin K$, and (b) $abM \nsubseteq N$ or $abM \nsubseteq K$. These reach to 4 cases:

(i) $am \notin N$ and $abM \nsubseteq N$

(ii) $am \notin N$ and $abM \nsubseteq K$

(iii) $am \notin K$ and $abM \nsubseteq N$

(iv) $am \notin K$ and $abM \nsubseteq K$.

First, we consider Case(i). Since $aRbRm \subseteq N \cap K \subseteq N$ and $am \notin N$, it follows from Proposition 2.25 that $bM \subseteq N$. This is a contradiction because $abM \nsubseteq N$. Hence Case(i) does not occur. Similarly, Case(iv) is not possible.

Next, Case(ii) is considered. Again, we obtain that $bM \subseteq N$ and then $bm \in N$. Let $r \in R$. Since $aRbRm \subseteq N \cap K \subseteq K$, it follows that $aR(brm) \subseteq aR(bRm) \subseteq K$. Hence $aM \subseteq K$ or $brm \in K$ because K is a prime R-ideal of M. If $aM \subseteq K$, then $abM \subseteq aM \subseteq K$ contradicts $abM \nsubseteq K$. Thus $brm \in K$. That is $bRm \subseteq K$. Since K is a prime R-ideal, $bM \subseteq K$ or $m \in K$. If $bM \subseteq K$, then $abM \subseteq K$ leading to the same contradiction. Therefore, $m \in K$ and then $bm \in K$. Hence $bm \in N \cap K$.

The proof of Case(iii) is similar to that of Case(ii).

Therefore, the intersection of each pair of prime R-ideals of M is a 2-absorbing R-ideal of M. $\square$

Next proposition shows the results of the intersection of an R-ideal and a prime (2-absorbing) R-ideal.

Proposition 3.2. *Let N and K be R-ideals of an R-module M with $K \nsubseteq N$.*

(1) If N is a prime R-ideal of M, then $K \cap N$ is a prime R-ideal of K; and

(2) If N is a 2-absorbing R-ideal of M, then $K \cap N$ is a 2-absorbing R-ideal of K.

Proof. We proof only (2) because the proof of (1) can be obtained similarly. Since N, K are R-ideals of M and $K \nsubseteq N$, it follows that $K \cap N$ is a proper R-ideal of K. Assume that N is a 2-absorbing R-ideal of M. Let $a, b \in R$ and $x \in K$ be such that $aRbRx \subseteq K \cap N$. Since K ia an R-ideal of M, we obtain that $abK \subseteq K$ and $ax, bx \in K$. Moreover, since $aRbRx \subseteq K \cap N \subseteq N$ and N is a 2-absorbing R-ideal of M, it follows that $abM \subseteq N$ or $ax \in N$ or $bx \in N$. Thus $abK \subseteq abK \cap abM \subseteq K \cap N$ or $ax \in K \cap N$ or $bx \in K \cap N$. Therefore, $K \cap N$ is a 2-absorbing R-ideal of K. $\square$

Next, we aim to consider prime R-ideals and 2-absorbing R-ideals of quotient modules over near rings.

Theorem 3.3. *Let N and K be R-ideals of an R-module M with $K \subseteq N$. Then*

(1) N is a prime R-ideal of M if and only if N/K is a prime R-ideal of M/K; and

(2) N is a 2-absorbing R-ideal of M if and only if N/K is a 2-absorbing R-ideal of M/K.

Proof. It suffices to proof only (2). First, assume that N is a 2-absorbing R-ideal of M. Then N/K is a proper R-ideal of M/K. Let $a, b \in R$ and $m \in M$ be such that $aRbR\,(m + K) \subseteq N/K$. Let $s, t \in R$. Thus $asbtm + K = asbt\,(m + K) \in aRbR\,(m + K) \subseteq N/K$. Then there exists $n \in N$ such that $asbtm + K = n + K$ so that $-n + asbtm \in K \subseteq N$ and then $asbtm \in N$. This shows that $aRbRm \subseteq N$. As a result, $am \in N$ or $bm \in N$ or $abM \subseteq N$ because N is a 2-absorbing R-ideal of M. Therefore, $a\,(m + K) \in N/K$ or $b\,(m + K) \in N/K$ or $ab(M/K) \subseteq N/K$. Hence N/K is a 2-absorbing R-ideal of M/K.

Conversely, assume that N/K is a 2-absorbing R-ideal of M/K. Then N is a proper R-ideal of M. Let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N$. Then $aRbR\,(m + K) \subseteq N/K$. Since N/K is a 2-absorbing R-ideal of M/K, we obtain that $a\,(m + K) \in N/K$ or $b\,(m + K) \in N/K$ or $ab(M/K) \subseteq N/K$. That is $am \in N$ or $bm \in N$ or $abM \subseteq N$. This implies that N is a 2-absorbing R-ideal of M. $\square$

Since a near ring R is also an R-module, 2-absorbing ideals of a near ring R are special cases of 2-absorbing R-ideals of the R-module R. Then all properties of 2-absorbing R-ideals of R-modules in this paper can be applied to 2-absorbing R-ideals of the near ring R. For example, we also obtain that "The intersection of each pair of prime ideals of a near ring R is a 2-absorbing ideal of R "as a corollary of Theorem 3.1.

In 2015, Chinwarakorn and Pianskool [3] introduced almost generalized 2-absorbing ideals of commutative rings with identities which is a generalization of 2-absorbing ideals of commutative rings with identities and investigated some properties of them on decomposable rings. This leads us to study some properties of 2-absorbing R-ideals of decomposable near rings.

Definition 3.4. A near ring R is said to be a **decomposable near ring** if it is a product of nonzero near rings equipped by componentwise addition and multiplication.

Example 3.5. Let $R_1 = (\{0, 1\}, +, \cdot)$ and $R_2 = (\{0, 1, a, b\}, +, \cdot)$ be the near rings given in Example 2.3 and Example 2.5, respectively. Then $R_1 \times R_2$ is a decomposable near ring.

Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable near ring and let M_i be an R_i-module for all $i \in \{1, 2, \ldots, n\}$. It is clear that the product of $M_1, M_2, \ldots, M_n$ is an R-module, i.e., $M_1 \times M_2 \times \cdots \times M_n$ is an R-module.

Proposition 3.6. *Let N_i be an R_i-ideal of an R_i-module M_i for all $i \in \{1, 2, \ldots, n\}$. Then $N_1 \times N_2 \times \cdots \times N_n$ is an R-ideal of M where $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$.*

Proof. The proof is straightforward. □

Next, some properties of 2-absorbing R-ideals on certain decomposable near rings are studied.

Lemma 3.7. *Let M_1 be an R_1-module, M_2 be an R_2-module, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(1) N_1 is a 2-absorbing (prime) R_1-ideal of M_1 if and only if $N_1 \times M_2$ is a 2-absorbing (prime) R-ideal of M; and

(2) N_2 is a 2-absorbing (prime) R_2-ideal of M_2 if and only if $M_1 \times N_2$ is a 2-absorbing (prime) R-ideal of M.

Proof. It suffices to prove only (1). First, assume that N_1 is a 2-absorbing R_1-ideal of M_1. Suppose that $(a, b)R(c, d)R(m_1, m_2) \subseteq N_1 \times M_2$ where $(a, b), (c, d) \in R$ and $(m_1, m_2) \in M$. Then $(aR_1cR_1m_1, bR_2dR_2m_2) = (a, b)R(c, d)R(m_1, m_2) \subseteq N_1 \times M_2$, i.e., $aR_1cR_1m_1 \subseteq N_1$ and $bR_2dR_2m_2 \subseteq M_2$. Since N_1 is a 2-absorbing R_1-ideal of M_1, it follows that $acM_1 \subseteq N_1$ or $am_1 \in N_1$ or $cm_1 \in N_1$. That is $(a, b)(c, d)M = (acM_1, bdM_2) \subseteq N_1 \times M_2$ or $(a, b)(m_1, m_2) = (am_1, bm_2) \in N_1 \times M_2$ or $(c, d)(m_1, m_2) = (cm_1, dm_2) \in N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a 2-absorbing R-ideal of M.

Conversely, assume that $N_1 \times M_2$ is a 2-absorbing R-ideal of M. Let $a, b \in R_1$ and $m_1 \in M_1$. Assume that $aR_1bR_1m_1 \subseteq N_1$. Let $x, y \in R_2$ and $m_2 \in M_2$. Then $(a, x)R(b, y)R(m_1, m_2) = (aR_1bR_1m_1, xR_2yR_2m_2) \subseteq N_1 \times M_2$. Since $N_1 \times M_2$ is a 2-absorbing R-ideal of M, it follows that $(a, x)(b, y)M \subseteq N_1 \times M_2$ or $(a, x)(m_1, m_2) \in N_1 \times M_2$ or $(b, y)(m_1, m_2) \in N_1 \times M_2$. Then $(abM_1, xyM_2) = (a, x)(b, y)M \subseteq N_1 \times M_2$ or $(am_1, xm_2) = (a, x)(m_1, m_2) \in N_1 \times M_2$ or $(bm_1, ym_2) = (b, y)(m_1, m_2) \in N_1 \times M_2$, i.e., $abM_1 \subseteq N_1$ or $am_1 \in N_1$ or $bm_1 \in N_1$. Therefore, N_1 is a 2-absorbing R_1-ideal of M_1. □

Theorem 3.8. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable near ring, M_i be an R_i-module, and N_i be an R_i-ideal of M_i for all $i \in \{1, 2, \ldots, n\}$. Then N_i is a 2-absorbing (prime) R_i-ideal of M_i if and only if $M_1 \times \cdots \times M_{i-1} \times N_i \times M_{i+1} \times \cdots \times M_n$ is a 2-absorbing (prime) R-ideal of $M_1 \times \cdots \times M_n$ for each $i \in \{1, 2, \ldots, n\}$.*

Proof. The result follows by applying Lemma 3.7. □

Recall that a near ring R is a module over itself. Moreover, if I is a prime (2-absorbing) ideal of a near ring R, then I is a prime (2-absorbing) R-ideal of the R-module R and vice versa.

Corollary 3.9. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable near ring and I_i be an ideal of R_i for all $i \in \{1, 2, \ldots, n\}$. Then I_i is a 2-absorbing (prime) ideal of R_i if and only if $R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_n$ is a 2-absorbing (prime) ideal of R for each $i \in \{1, 2, \ldots, n\}$.*

Next theorem shows a condition that makes $N_1 \times N_2$ be a 2-absorbing $(R_1 \times R_2)$-ideal of an $(R_1 \times R_2)$-module $M_1 \times M_2$ where each N_i is a proper R_i-ideal of M_i.

Theorem 3.10. *If N_1 is a prime R_1-ideal of an R_1-module M_1 and N_2 is a prime R_2-ideal of an R_2-module M_2, then $N_1 \times N_2$ is a 2-absorbing R-ideal of the R-module M where $R = R_1 \times R_2$ and $M = M_1 \times M_2$.*

Proof. Assume that N_1 is a prime R_1-ideal of an R_1-module M_1 and N_2 is a prime R_2-ideal of an R_2-module M_2. Then $N_1 \times N_2$ is a proper R-ideal of M. Let $(a,b),(c,d) \in R_1 \times R_2$ and $(m_1,m_2) \in M_1 \times M_2$. Assume that $(a,b)R(c,d)R(m_1,m_2) \subseteq N_1 \times N_2$ but $(a,b)(c,d)M \nsubseteq N_1 \times N_2$ and $(a,b)(m_1,m_2) \notin N_1 \times N_2$. Then we can conclude that (a) $am_1 \notin N_1$ or $am_2 \notin N_2$, and (b) $acM_1 \nsubseteq N_1$ or $bdM_2 \nsubseteq N_2$. There are 4 cases to be considered:

(i) $am_1 \notin N_1$ and $acM_1 \nsubseteq N_1$
(ii) $am_2 \notin N_2$ and $bdM_2 \nsubseteq N_2$
(iii) $am_1 \notin N_1$ and $bdM_2 \nsubseteq N_2$
(iv) $am_2 \notin N_2$ and $acM_1 \nsubseteq N_1$.

We claim $(c,d)(m_1,m_2) \in N_1 \times N_2$. First, we consider Case(i). Note that $aR_1cR_1m_1 \subseteq N_1$ and $bR_2dR_2m_2 \subseteq N_2$ because $(aR_1cR_1m_1, bR_2dR_2m_2) = (a,b)R(c,d)R(m_1,m_2) \subseteq N_1 \times N_2$. Since N_1 is a prime R_1-ideal of M_1 and $am_1 \notin N_1$, we obtain from Proposition 2.25 that $cM_1 \subseteq N_1$ so that $acM_1 \subseteq N_1$ which is a contradiction. Then Case(i) is not possible. In addition, Case(ii) is absurd.

Next, Case (iii) is considered. Similarly, $cM_1 \subseteq N_1$. Thus $cm_1 \in N_1$. Moreover, $bR_2dR_2m_2 \subseteq N_2$. Let $r \in R_2$. Then $bR_2drm_2 \subseteq N_2$. Since N_2 is a prime R_2-ideal of M_2, we have $bM_2 \subseteq N_2$ or $drm_2 \in N_2$. If $bM_2 \subseteq N_2$, then $bdM_2 \subseteq bM_2 \subseteq N_2$ contradicts $bdM_2 \nsubseteq N_2$. Then $drm_2 \in N_2$. That is $dR_2m_2 \subseteq N_2$. And again, since N_2 is a prime R_2-ideal of M_2 and $bdM_2 \nsubseteq N_2$, we get that $m_2 \in N_2$ so that $dm_2 \in N_2$. Therefore, $(c,d)(m_1,m_2) = (cm_1, dm_2) \in N_1 \times N_2$.

The proof of Case(iv) is similar to that of Case(iii). Hence $N_1 \times N_2$ is a 2-absorbing R-ideal of M. $\square$

However, it is not necessary true that the product of prime R-ideals is a prime R-ideal. For example, let $N_1 = \{0\}$ be the prime R_1-ideal of $M_1 = \{0,1\}$ and $N_2 = \{0,b\}$ be the prime R_2-ideal of $M_2 = \{0,1,a,b\}$ given in Example 2.21 and Example 2.22, respectively. Let $R = R_1 \times R_2$. Then $N_1 \times N_2$ is not a prime R-ideal of $M_1 \times M_2$ because $(0,a)R(1,b) \subseteq \{(0,0),(0,b)\} = N_1 \times N_2$ but $(0,a),(1,b) \notin N_1 \times N_2$.

We obtain from Theorem 3.10 that $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$ where I_1 and I_2 are prime ideals of the near rings R_1 and R_2, respectively. Moreover, if R_1 and R_2 are zero symmetric near rings, then the converse of Theorem 3.10 is true.

Theorem 3.11. *Let R_1 and R_2 be zero symmetric near rings with identities, I_1 and I_2 be proper ideals of R_1 and R_2, respectively. Then I_1 is a prime ideal of R_1 and I_2 is a prime ideal of R_2 if and only if $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$.*

Proof. To prove the sufficient part, assume that $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$. Let $a,b \in R_1$ and $x,y \in R_2$. Suppose that $aR_1b \subseteq I_1$ and $xR_2y \subseteq I_2$. Then $aR_1 1 R_1 b \subseteq I_1$ and $xy = x1y \in xR_2y \subseteq I_2$. Since I_2 is an ideal of R_2 and $xy \in I_2$, we obtain that $xyR_2 \subseteq I_2$. Moreover, $R_2xyR_2 \subseteq RI_2 \subseteq I_2$ because R_2 is a zero symmetric near ring. Note that $(a,1)R(1,xy)R(b,1) = (aR_1 1 R_1 b, 1R_2xyR_2 1) \subseteq I_1 \times I_2$. Since $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$, it follows that $(a,1)(1,xy) \in I_1 \times I_2$ or $(1,xy)(b,1) \in I_1 \times I_2$ or $(a,1)(b,1) \in I_1 \times I_2$, i.e., $(a,xy) \in I_1 \times I_2$ or $(b,xy) \in I_1 \times I_2$ or $(ab,1_2) \in I_1 \times I_2$. But I_2 is a proper ideal of R_2 so that $(ab,1) \in I_1 \times I_2$ is not possible. Hence $(a,xy) \in I_1 \times I_2$ or $(b,xy) \in I_1 \times I_2$. Thus $a \in I_1$ or $b \in I_1$. Therefore, I_1 is a 2-absorbing ideal of R_1. Similarly, we obtain that I_2 is a 2-absorbing ideal of R_2. $\square$

The last result provides a characterization of being a 2-absorbing ideal of the ideal $I_1 \times I_2 \times I_3$ of a decomposable near ring where I_1 is proper.

Theorem 3.12. *Let $R = R_1 \times R_2 \times R_3$ where R_1, R_2 and R_3 are zero symmetric near rings with identities, I_1 be a proper ideal of R_1, I_2 and I_3 be ideals of R_2 and R_3, respectively. Then the following statements are equivalent.*

(1) $I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of R.

(2) I_1 is 2-absorbing ideal of R_1, $I_2 = R_2$ and $I_3 = R_3$ or
I_1, I_2 are prime ideals and $I_3 = R_3$ or
I_1, I_3 are prime ideals and $I_2 = R_2$.

Proof. First, assume that $I := I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of R. Then I is a nonempty subset of R. Let $(a, b, c) \in I$. Note that $(a, 1, 1)R(1, b, 1)R(1, 1, c) = (aR_1, R_2bR_2, R_3c) \subseteq I_1 \times I_2 \times I_3 = I$ because I_1, I_2 and I_3 are ideals of zero symmetric near rings. Since I is a 2-absorbing ideal of R, $(a, 1, 1)(1, b, 1) \in I$ or $(1, b, 1)(1, 1, c) \in I$ or $(a, 1, 1)(1, 1, c) \in I$, i.e., $(a, b, 1) \in I$ or $(1, b, c) \in I$ or $(a, 1, c) \in I$. Then $I_3 = R_3$ or $I_1 = R_1$ or $I_2 = R_2$. But I_1 is a proper ideal of R_1, it follows that $I_3 = R_3$ or $I_2 = R_2$. This reaches to 3 cases:

(i) $I_2 = R_2$ and $I_3 = R_3$,

(ii) $I_2 \neq R_2$ or $I_3 = R_3$,

(iii) $I_2 = R_2$ or $I_3 \neq R_3$.

The first case leads to the result that $I = I_1 \times (R_2 \times R_3)$ where I_1 is 2-absorbing ideal of R_1 by Corollary 3.9. Next, we proof the second case by showing that I_1 and I_2 are prime ideals. Let $a, b \in R_1$ and $x, y \in R_2$. Assume that $aR_1b \subseteq I_1$ and $xR_2y \subseteq I_2$. Then $(a, 1, 1)R(1, xy, 1)R(b, 1, 1) = (aR_1b, R_2xyR_2, R_3) \subseteq I_1 \times I_2 \times I_3 = I$ because I_2 is an ideal of the zero symmetric near ring R_2. Since I ia a 2-absorbing ideal of R, $(a, 1, 1)(1, xy, 1) \in I$ or $(1, xy, 1)(b, 1, 1) \in I$ or $(a, 1, 1)(b, 1, 1) \in I$, i.e., $(a, xy, 1) \in I$ or $(b, xy, 1) \in I$ or $(ab, 1, 1) \in I$. Since $I_2 \neq R_2$, it follows that $(a, xy, 1) \in I$ or $(b, xy, 1) \in I$. That is $a \in I_1$ or $b \in I_1$. Therefore, I_1 is a prime ideal of R_1. Similarly, we obtain that I_2 is a prime ideal of R_2. The proof of Case(iii) is similar to that of Case(ii).

Conversely, if $I = I_1 \times R_2 \times R_3$ and I_1 is a 2-absorbing ideal of R_1, then I is a 2-absorbing ideal of R by Corollary 3.9 because $R_2 \times R_3$ is a near ring. Consider Case(ii), since I_1 and I_2 are prime ideals, $I_1 \times I_2$ is a 2-absorbing ideal by Theorem 3.11. It is easy to verify that I is a 2-absorbing ideal of R by Corollary 3.9 again. The last case is similar to the previous case. $\square$

By the results of Theorem 3.11 and Theorem 3.12, we can see that, in order to obtain these results, being a zero symmetric near ring with identity is crucial.

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