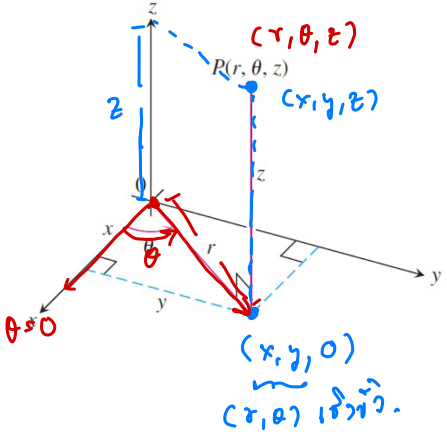
อินทิกรัลสามชั้นในพิกัดทรงกระบอก (Triple Integrals in cylindrical coordinates)

ทบทวนพิกัดทรงกระบอก :

เปลี่ยน $(x, y, z) \to (r, \theta, z)$



ความสัมพันธ์ :

- $x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$
- $r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$

$dV = dx\,dy\,dz$

อินทิกรัล : ฟังก์ชัน $\iiint f(x,y,z)\,dV(x,y,z)$

$E(x,y,z)$

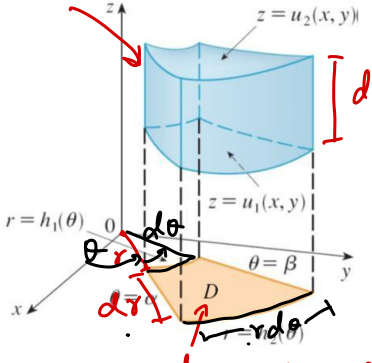
$\Downarrow$ $F(r,\theta,z)$

$$\iiint f(r\cos\theta, r\sin\theta, z)\, dV(r,\theta,z)$$

$E(r,\theta,z)$

$dV = dz\,dA(r,\theta) = r\,dz\,dr\,d\theta$

$dV = r\,dr\,d\theta$.



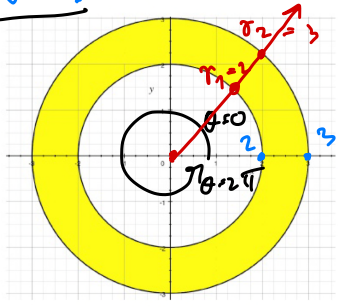
$dA(r,\theta) \Rightarrow r\,dr\,d\theta$

$$\Rightarrow \iint_{R(r,\theta)} \left(\int_z f(r\cos\theta, r\sin\theta, z)\, dz \right) dV \; r\,dr\,d\theta$$

$E(r,\theta,z)$

ตัวอย่าง 6.3.1 จงหาค่าของ $\iiint_E y\,dV$ เมื่อ E คือทรงตันที่ปิดด้านบนด้วยระนาบ $z = x + 2$ อยู่เหนือระนาบ XY และอยู่ระหว่างทรงกระบอก $x^2 + y^2 = 4$ และ $x^2 + y^2 = 9$

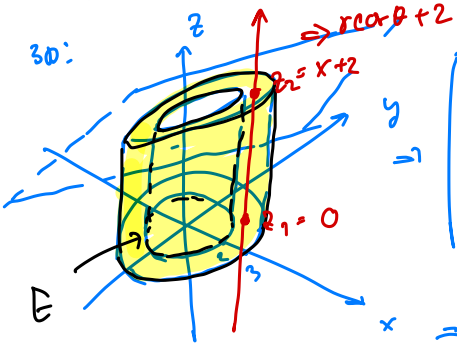
proj. XY



2^2 3^2

$$\iiint_E y\,dV \Rightarrow \int\int_{R(r,\theta)} \left(\int_z r\sin\theta\, dz \right) r\,dr\,d\theta$$

3D:



$$\Rightarrow \int_{\theta=0}^{2\pi}\int_{r=2}^{3}\left(\int_{z=0}^{z=r\cos\theta+2} (r\sin\theta)\cdot r\,dz\right) dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{2\pi}\int_{r=2}^{3} r^2\sin\theta\,(z)\Big|_{z=0}^{z=r\cos\theta+2}\, dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{2\pi}\int_{r=2}^{3} (r^2\sin\theta)\left[(r\cos\theta+2) - 0\right] dr\,d\theta.$$

$$\Rightarrow \int_{\theta=0}^{2\pi}\int_{r=2}^{3} r^3\sin\theta\cos\theta + 2r^2\sin\theta\, dr\,d\theta$$

$u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$\Rightarrow \int_{\theta=0}^{2\pi} (\sin\theta\cos\theta)\left(\frac{r^4}{4}\right)\Big|_{r=2}^{r=3} + (2\sin\theta)\left(\frac{r^3}{3}\right)\Big|_{r=2}^{r=3} d\theta$$

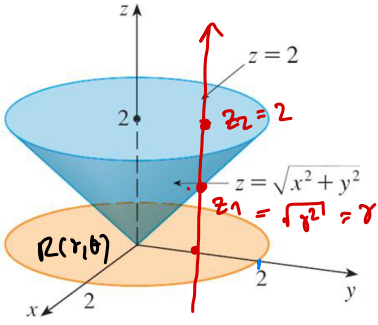
$$\Rightarrow \left[\frac{3^4-2^4}{4}\right]\left(\frac{\sin^2\theta}{2}\right)\Big|_{\theta=0}^{\theta=2\pi} + 2\left[\frac{3^3-2^3}{3}\right](-\cos\theta)\Big|_{\theta=0}^{\theta=2\pi}$$

$= \ldots 0$ ■

ตัวอย่าง 6.3.2 จงหาค่าของ $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

$dV = r\,dz\,dr\,d\theta$

3D:

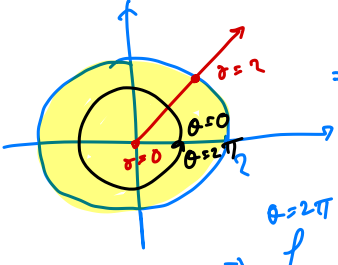


$f(x,y,z) \Rightarrow f(r\cos\theta, r\sin\theta, z)$

$$\int_{R(r,\theta)}\int_{z=r}^{z=2}\underbrace{(r\cos\theta)^2+(r\sin\theta)^2}_{r^2(\cos^2\theta+\sin^2\theta)=1}\,\underbrace{r\,dz\,dr\,d\theta}_{dV}$$

$$\Rightarrow \int\int\left(\int_{z=r}^{z=2} r^2\cdot r\,dz\right)dr\,d\theta$$

proj. XY



$$\Rightarrow \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z=r}^{z=2} r^3\,dz\,dr\,d\theta$$

$$\Rightarrow \int_{0}^{2\pi}\int_{0}^{2} r^3(z)\Big|_{z=r}^{z=2}\,dr\,d\theta$$

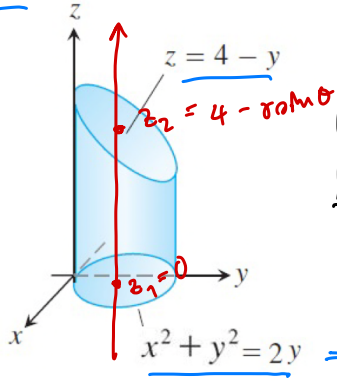
$$\Rightarrow \int_{0}^{2\pi}\int_{0}^{2} r^3(2-r)\,dr\,d\theta$$

$$\Rightarrow \int_{0}^{2\pi}\left[2\frac{r^4}{4}-\frac{r^5}{5}\right]\Big|_{0}^{2}\,d\theta$$

$$\Rightarrow \left[\left(\frac{2\cdot 2^4}{4}-\frac{2^5}{5}\right)-0\right](\theta)\Big|_{0}^{2\pi} = 2\pi\cdot 2^5\cdot\left(\frac{1}{4}-\frac{1}{5}\right)$$

$$= \frac{\pi\cdot 2^6}{20}$$

ตัวอย่าง 6.3.3 จงเขียน $\iiint_E f(x,y,z)dV$ ในรูปอินทิกรัลสามชั้นในพิกัดทรงกระบอก โดยไม่ต้องคำนวณค่า เมื่อ E คือทรงตันที่ปิดด้านบนด้วยระนาบ $z = 4 - y$ และอยู่ภายในทรงกระบอก $x^2 + y^2 = 2y$



3D:

$z_2 = 4 - r\sin\theta$

$z_1 = 0$

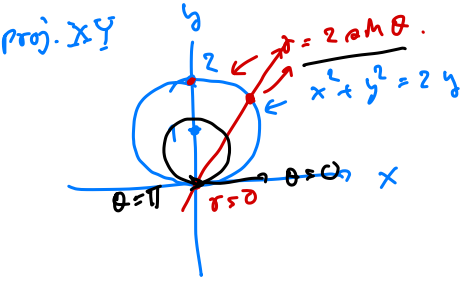
$x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=0}^{4-r\sin\theta} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

dV

$R(r,\theta)$

proj. xy



$r = 2\sin\theta$

$x^2 + y^2 = 2y$

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 2r\sin\theta$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2r\sin\theta$

$= 1$

$\Rightarrow r = 2\sin\theta$