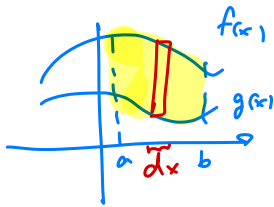


Định lý: Tính diện tích 2 chiều

đơn vị 1: $\iint_R f(x,y) dA$

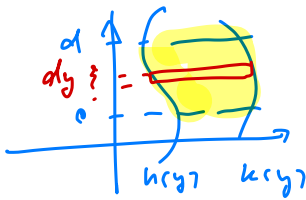
⇒ đơn vị 1: dx đơn vị $\Rightarrow dA = dy dx$



$x=b$ $y=f(x)$
 $\Rightarrow \iint_{x=a, y=g(x)} f(x,y) dy dx$
 $x=a$ $y=g(x)$

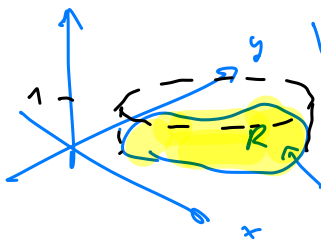
đơn vị $dx dy$
 luôn luôn đúng!
 nếu, nếu không
R (khi đó luôn đúng)
 luôn luôn đúng

⇒ đơn vị 2: dy đơn vị $\Rightarrow dA = dx dy$



$y=d$ $x=k(y)$
 $\Rightarrow \iint_{y=c, x=h(y)} f(x,y) dx dy$
 $y=c$ $x=h(y)$

⇒ đơn vị 3: luôn luôn đúng đơn vị 2 chiều



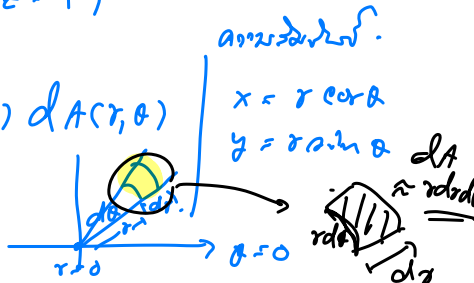
$V = \iint_R 1 dA$

đơn vị 3 chiều (R=1) ở 2 chiều tính diện tích

⇒ đơn vị 3 chiều tính diện tích

$\iint_{R(x,y)} f(x,y) dA(x,y) \Rightarrow \iint_{R(r,\theta)} f(r,\theta) dA(r,\theta)$

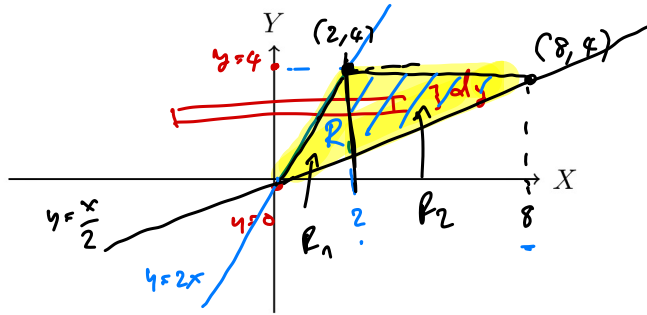
$dA(r,\theta) = r dr d\theta$



8. กำหนดอินทิกรัลสองชั้นในระบบพิกัดฉาก

$$I = \iint_R f(x, y) \, dA = \int_0^4 \int_{y/2}^{2y} \cos(x^2) \, dx \, dy \quad \begin{matrix} \rightarrow x=2y \Rightarrow y = \frac{x}{2} \\ y = \frac{x}{2} \Rightarrow x = 2y \Rightarrow y = 2x \end{matrix}$$

8.1. จงวาดรูปและแรเงาบริเวณ R ที่สอดคล้องกับ I (ระบุจุดตัดให้ชัดเจน)



8.2. จงหาค่าของ

$$\int_0^4 \int_{y/2}^{2y} \cos(x^2) \, dx \, dy \quad \Rightarrow \iint_R \cos(x^2) \, dy \, dx =$$

$$\Rightarrow \iint_{R_1} \cos(x^2) \, dy \, dx + \iint_{R_2} \cos(x^2) \, dy \, dx$$

$x=0 \quad y=\frac{x}{2}$ $x=2 \quad y=\frac{x}{2}$ $x=8 \quad y=4$
 (R_1) (R_2)

$$\Rightarrow \int_{x=0}^{x=2} \cos(x^2) y \Big|_{y=\frac{x}{2}}^{y=2x} \, dx + \int_{x=2}^{x=8} \cos(x^2) y \Big|_{y=\frac{x}{2}}^{y=4} \, dx$$

$$\Rightarrow \int_{x=0}^{x=2} \cos(x^2) \left(2x - \frac{x}{2}\right) \, dx + \int_{x=2}^{x=8} \cos(x^2) \left(4 - \frac{x}{2}\right) \, dx$$

① ② ③

$$\Rightarrow \int_{x=0}^{x=2} \cos(x^2) \left(\frac{3x}{2}\right) \, dx + \int_{x=2}^{x=8} \cos(x^2) \left(-\frac{x}{2}\right) \, dx + \int_{x=2}^{x=8} \cos(x^2) 4 \, dx$$

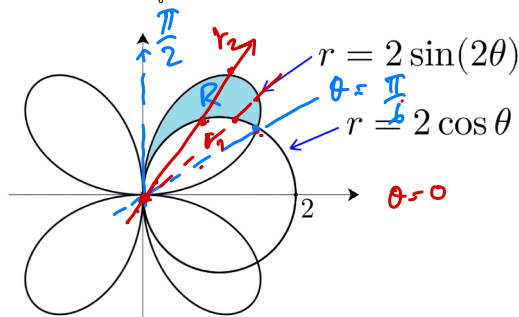
$u = x^2 \Rightarrow du = 2x \, dx$ $u = x^2$

✓

$$\int u dv = uv - \int v du$$

$$\begin{aligned} \textcircled{3} : 4 \int \underbrace{\cos(x^2)}_u \underbrace{dx}_v &= 4 \left[x \cos(x^2) - \int x \underbrace{d(\cos(x^2))}_{-\sin(x^2)(2x) dx} \right] \\ &= 4 \left[x \cos(x^2) + \int 2x^2 \sin(x^2) dx \right] \end{aligned}$$

9. กำหนดบริเวณ R เป็นบริเวณที่แรเงาดังรูป



จงหาพื้นที่ A $\left(\sin(2A) = 2 \sin A \cos A, \sin^2 A = \frac{1 - \cos(2A)}{2} \text{ และ } \cos^2 A = \frac{1 + \cos(2A)}{2} \right)$

$$\begin{aligned} R &= \iint_R 1 \, dA = \int_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}} \int_{r = 2 \cos \theta}^{r = 2 \sin(2\theta)} \underbrace{1}_{\text{area element}} \, r \, dr \, d\theta \\ &= \int_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}} \left. \frac{r^2}{2} \right|_{r = 2 \cos \theta}^{r = 2 \sin(2\theta)} d\theta \\ &= \int_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}} \left[\frac{(2 \sin(2\theta))^2}{2} \right] - \left[\frac{(2 \cos \theta)^2}{2} \right] d\theta \\ &= \int_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}} 2 \sin^2(2\theta) - 2 \cos^2 \theta \, d\theta. \end{aligned}$$

$$\left[\begin{aligned} \sin^2 \theta &= \frac{1 - \cos(2\theta)}{2} \\ \cos^2 \theta &= \frac{1 + \cos(2\theta)}{2} \end{aligned} \right] = \int_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}} 2 \left[\frac{1 - \cos(4\theta)}{2} \right] - 2 \left[\frac{1 + \cos(2\theta)}{2} \right] d\theta$$

$$= \int_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}} -\cos(4\theta) - \cos(2\theta) d\theta$$

$$= \left[-\frac{\sin(4\theta)}{4} - \frac{\sin(2\theta)}{2} \right] \Bigg|_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{2}}$$

$$= \left[-\frac{\sin\left(\frac{4\pi}{2}\right)}{4} - \frac{\sin\left(\frac{2\pi}{2}\right)}{2} \right] + \left[\frac{\sin\left(\frac{4\pi}{6}\right)}{4} + \frac{\sin\left(\frac{2\pi}{6}\right)}{2} \right]$$

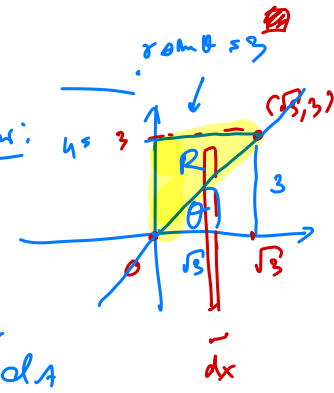
$$= 0$$

10. กำหนดให้อินทิกรัลสองชั้นในระบบพิกัดฉาก

ถามวิธีแก้:
 $x = r \cos \theta$
 $y = r \sin \theta$

$$I = \int_{x=0}^{x=\sqrt{3}} \int_{y=\sqrt{3}x}^{y=3} e^{x+y} dy dx$$

(R) $f(x,y) dA$

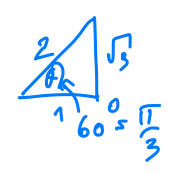


จงเขียน I ในรูปอินทิกรัลสองชั้นในระบบพิกัดเชิงขั้ว (โดยไม่ต้องคำนวณค่า)

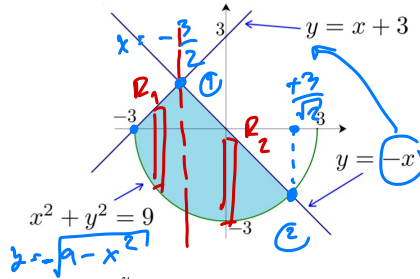
$$I = \int_{r=0}^{r=3 \sec \theta} \int_{\theta=0}^{\theta=\frac{\pi}{3}} e^{r \cos \theta + r \sin \theta} r dr d\theta$$

(1) (2) (3)

$\tan \theta = \frac{3}{\sqrt{3}}$
 $\tan \theta = \sqrt{3}$



11. กำหนดให้ R เป็นบริเวณที่แรเงาดังรูป



$$x \text{ n. } R = \iint_R 1 \, dA$$

จุดตัด: ①

$$-x = x + 3$$

$$-2x = 3 \Rightarrow x = -\frac{3}{2}$$

จุดตัด: ②:

$$x^2 + (-x)^2 = 9$$

$$2x^2 = 9$$

$$x = \pm \sqrt{\frac{9}{2}} = \pm \frac{3}{\sqrt{2}}$$

จงเขียนพื้นที่ของ R ในระบบพิกัดต่อไปนี้ (โดยไม่ต้องคำนวณค่า)

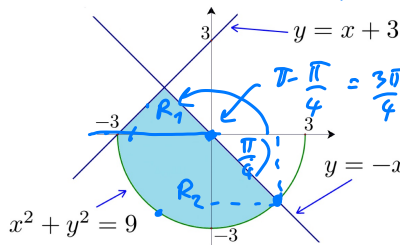
11.1. ระบบพิกัดฉาก $R =$ _____

11.2. ระบบพิกัดเชิงขั้ว $R =$ _____

$$11.1: \int_{x=-3}^{x=-\frac{3}{2}} \int_{y=-\sqrt{9-x^2}}^{y=x+3} 1 \, dy \, dx + \int_{x=-\frac{3}{2}}^{x=\frac{3}{2}} \int_{y=-\sqrt{9-x^2}}^{y=-x} 1 \, dy \, dx$$

(R_1) (R_2)

บริเวณที่แรเงาดังรูป



$$\Rightarrow r \sin \theta = r \cos \theta + 3$$

$$\Rightarrow r(\sin \theta - \cos \theta) = 3$$

$$\Rightarrow r = \frac{3}{\sin \theta - \cos \theta}$$

$$\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$$

$$\theta = \frac{2\pi}{4} \quad r = 3$$

$$+ \iint 1 \, r \, dr \, d\theta$$

$$\theta = \pi \quad r = 0$$

11.2 นวัตกรรมขั้ว

$$\int_{\theta = \frac{3\pi}{4}}^{\theta = \pi} \int_{r=0}^{r=\frac{3}{\sin \theta - \cos \theta}} 1 \, r \, dr \, d\theta$$

(R_1)

(R_2)

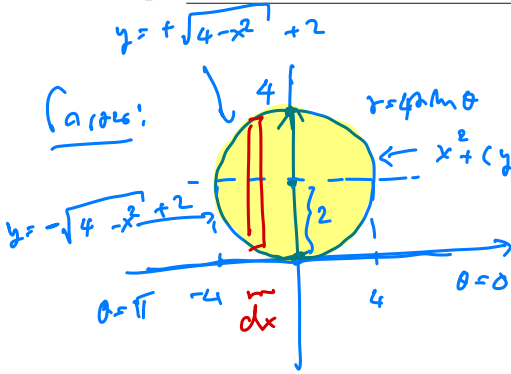
12. กำหนดให้อินทิกรัลสองชั้นในระบบพิกัดเชิงขั้ว

$$I = \int_0^{\pi} \int_0^{4 \sin \theta} r^3 \sin(r^2) dr d\theta$$

$$\Rightarrow \int_R \int f(r, \theta) dA(r, \theta)$$

จงเขียน I ในรูปอินทิกรัลสองชั้นในระบบพิกัดฉาก (โดยไม่ต้องคำนวณค่า)

$$I = \underline{\hspace{10cm}}$$



$$\theta = 4 \sin \theta$$

$$x^2 + (y-2)^2 = 2 \quad \text{วงกลม}$$

$$x = 4 \quad y = +\sqrt{4-x^2} + 2$$

$$\int \int (x^2 + y^2) \sin(x^2 + y^2) dy dx$$

$$x = -4 \quad y = -\sqrt{4-x^2} + 2 \quad dA$$