The 22nd Annual Meeting in Mathematics (AMM 2017)
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Benford's law for chains of truncated distributions

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Abstract

It is quite prevalent that the first digits of real world data are distributed approximately according to a discrete logarithmic distribution proposed and studied by Benford, hence the name Benford's law. Given an initial distribution $F = F_1$, we study a sequence of random variables X_n's, or equivalently distributions F_n's, for which X_{n+1} is distributed according to F, right-truncated by X_n. The sequence is called the chain of truncated distributions generated by F. We show that if F is supported on $[0, k]$, $k > 0$ and uniformly distributed on a neighborhood of 0 then the chain of truncated distributions generated by F satisfies Benford's law in the limit.

Keywords: Benford's law, first significant digits, truncated distributions, chains of truncated distributions.

2010 MSC: Primary 11F06; Secondary 60A10, 60F99.

1 Introduction

The first digit of a number is the leading significant digit appearing on the left excluding 0 and signs, e.g. the first digit of 13.467 is 1, the first digit of 0.002456 is 2. Some might expect the probability of occurrence of first significant digits to be quite uniform. But it is not true for many real world data and many sequences of random variables modeling processes in nature.

In 1930s [1], F. Benford studied the first digit of many data in the world, e.g. river areas and population, and discovered that, in most data, the frequency of the first digit being 1 is much higher than those of $2, 3, ..., 8$ and 9. In fact, he claimed that the probability that the first digit is d equals $\log_{10}\left(\frac{1+d}{d}\right)$, for $d = 1, 2, 3, ..., 9$.

In 2006 [3], A.E. Kossovsky considered a chain of distributions, which is a sequence of random variables $X_i, i \in \mathbb{I}$, all of whose distributions are members of a parametric family of distributions, such that X_i is the parameter of the distribution of X_{i+1}. He conjectured that the distributions of the leading digits of a chain of probability distributions converge to Benford's law. In 2008 [2], D. Jang et al. obtained a set of sufficient conditions for chains of distributions to satisfy Benford's law in the limit.

Here, we consider chains of truncated distributions, defined as chains of distributions whose parameters are the right end points of the supports. More precisely, we will find a class of

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distributions whose chains of truncated distributions satisfy Benford's law in the limit. Section 2 contains necessary mathematical backgrounds for the manuscript. In section 3, we state and prove some results leading to our main result saying that chains of truncated distributions generated by a continuous distribution on $[0, k]$, $k > 0$ which is uniformly distributed on a neighborhood of 0 satisfy Benford's law in the limit.

2 Preliminaries

For a real number x, we may write $x = M \times 10^k$, where $k \in \mathbb{N}$ and $1 \leq M < 10$ is the mantissa of x. Let $\mathcal{D}$ be a distribution supported on $[0, k]$ for $k > 0$. A random variable $X \sim \mathcal{D}$ is said to satisfy Benford's law if the probability of its mantissa lying between 1 and s is $\log_{10} s$, for $s = 2, 3, 4, \ldots, 10$. A sequence of random variables X_n is said to satisfy Benford's law in the limit if the probability of mantissa of X_n lying between 1 and s converges to $\log_{10} s$, i.e.,

$$\lim_{n\to\infty} \sum_{l=-\infty}^{\infty} \mathbb{P}\left(10^{-l} \leq X_n < s10^{-l}\right) = \log_{10} s,$$

where $\mathbb{P}$ is the probability function.

Let f be a function on $\mathbb{R}$. We say that f tends to 0 rapidly at infinity if for each positive integer m the function $x \mapsto |x|^m f(x)$, for $x \in \mathbb{R}$, is bounded for $|x|$ sufficiently large. Define the Schwartz class to be the set of functions on $\mathbb{R}$ whose derivatives of all orders are infinitely differentiable and which tend to 0 rapidly at infinity.

Let $f \in L^1(\mathbb{R})$. Define the Fourier transform $\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi itx}\, dx$ for $t \in \mathbb{R}$. The Poisson Summation Formula states that if f is in Schwartz class, then $\sum_{l\in\mathbb{Z}} f(l) = \sum_{t\in\mathbb{Z}} \hat{f}(t)$, where $\hat{f}$ is the Fourier transform of f. See [4].

3 Chains of Truncated Distributions

Definition 3.1. Let $\mathcal{D}[0, k]$ be a distribution supported on $[0, k]$, $k > 0$, with cumulative distribution function F such that $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. For $a \in (0, k]$, the distribution $\mathcal{D}[0, a]$ is supported on $[0, a]$ with cumulative distribution function F_a defined by $F_a(x) = \frac{F(x)}{F(a)}$ for $x \in [0, a]$ and $F_a(x) = 1$ for $x > a$. Define $X_1 \sim \mathcal{D}[0, k]$ and $X_n \sim \mathcal{D}[0, X_{n-1}]$ for $n \geq 2$. The chain of distributions of X_n is called a chain of truncated distributions generated by $\mathcal{D}[0, k]$.

Lemma 3.2. *Let $\mathcal{D}[0, k]$ be a distribution on $[0, k]$ with the probability density function f and the cumulative distribution function F. If X_n is a chain of truncated distributions generated by $\mathcal{D}[0, k]$, then for each $n \geq 2$,*

$$f_{X_n}(x_n) = \frac{f(x_n)}{(n-1)!}\left(\ln\left(\frac{1}{F(x_n)}\right)\right)^{n-1}, 0 < x_n < k, \tag{3.1}$$

and

$$F_{X_n}(x_n) = F_{X_{n-1}}(x_n) + \frac{F(x_n)}{(n-1)!}\left(\ln\left(\frac{1}{F(x_n)}\right)\right)^{n-1}, 0 < x_n < k, \tag{3.2}$$

where f_{X_n} denotes the probability density function of X_n and F_{X_n} denotes the distribution function of X_n.

Proof. We will prove the above equations by induction. Firstly, $f_{X_1}(x) = f(x)$, $F_{X_1}(x) = F(x)$.

Then

$$\begin{aligned} F_{X_2}(x_2) &= \int_0^k P(X_2 \le x_2 | X_1 = t)\, dF_{X_1}(t) \\ &= \int_0^{x_2} P(X_2 \le x_2 | X_1 = t)\, dF_{X_1}(t) + \int_{x_2}^k P(X_2 \le x_2 | X_1 = t)\, dF_{X_1}(t) \\ &= \int_0^{x_2} 1\, dF_{X_1}(t) + \int_{x_2}^k \frac{F(x_2)}{F(t)}\, dF(t) \\ &= F_{X_1}(x_2) + F(x_2) \ln\left(\frac{1}{F(x_2)}\right). \end{aligned}$$

Then $f_{X_2}(x_2) = \dfrac{d}{dx_2} F_{X_2}(x_2) = f_{X_1}(x_2) - f(x_2) + f(x_2)\ln\left(\dfrac{1}{F(x_2)}\right) = f(x_2)\ln\left(\dfrac{1}{F(x_2)}\right)$.
Let $n \in \mathbb{N}$ be such that (3.1) and (3.2) holds. Then

$$\begin{aligned} F_{X_{n+1}}(x_{n+1}) &= \int_0^k P(X_{n+1} \le x_{n+1} | X_n = t)\, dF_{X_n}(t) \\ &= \int_0^{x_{n+1}} P(X_{n+1} \le x_{n+1} | X_n = t)\, dF_{X_n}(t) + \int_{x_{n+1}}^k P(X_{n+1} \le x_{n+1} | X_n = t)\, dF_{X_n}(t) \\ &= \int_0^{x_{n+1}} 1\, dF_{X_n}(t) + \int_{x_{n+1}}^k \frac{F(x_{n+1})}{F(t)} \frac{f(t)}{(n-1)!} \left(\ln\left(\frac{1}{F(t)}\right)\right)^{n-1} dt \\ &= F_{X_n}(x_{n+1}) + \frac{F(x_{n+1})}{n!}\left(\ln\left(\frac{1}{F(x_{n+1})}\right)\right)^n. \end{aligned}$$

And hence

$$\begin{aligned} f_{X_{n+1}}(x_{n+1}) &= \frac{d}{dx_{n+1}} F_{X_{n+1}}(x_{n+1}) \\ &= f_{X_n}(x_{n+1}) - \frac{f(x_{n+1})}{(n-1)!}\left(\ln\left(\frac{1}{F(x_{n+1})}\right)\right)^{n-1} + \frac{f(x_{n+1})}{n!}\left(\ln\left(\frac{1}{F(x_{n+1})}\right)\right)^n \\ &= \frac{f(x_{n+1})}{n!}\left(\ln\left(\frac{1}{Fx(_{n+1})}\right)\right)^n. \end{aligned}$$

By induction, the proof is complete. □

Theorem 3.3. *Let a and k be positive numbers not necessarily equal. Let $\mathcal{D}[0,a]$ be a distribution supported on $[0,a]$ with the probability density f and whose chain of truncated distributions satisfies Benford's law in the limit. Given a distribution $\bar{\mathcal{D}}[0,k]$ with the probability density h defined by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in [0,\varepsilon), \\ g(x), & \text{if } x \in [\varepsilon, k], \end{cases}$$

for some integrable function g and $\varepsilon \in (0,k]$, the chain of truncated distributions generated by $\bar{\mathcal{D}}$ satisfies Benford's law in the limit.

Proof. Let X_n be a chain of truncated distribution with X_1 having a density h and s an integer in $[2,10]$. Then, by Lemma 3.2, the density of X_n is $h_n(x_n) = \dfrac{h(x_n)}{(n-1)!}\left(\ln\left(\dfrac{1}{H(x_n)}\right)\right)^{n-1}$ for all $n \ge 2$. Let $m \in \mathbb{Z}$ be such that $10^{-m+1} < \varepsilon$, δ the minimum of $\{x \in \mathbb{Z} : 10^{-x} < k\}$ and υ the minimum of $\{x \in \mathbb{Z} : 10^{-x} < a\}$. Let Y_n be a chain of truncated distribution with $Y_1 \sim \mathcal{D}[0,a]$. Then $\lim\limits_{n\to\infty} \sum\limits_{l=-\infty}^{\infty} \mathbb{P}\left(10^{-l} \le Y_n < s10^{-l}\right) = \log_{10} s$. Then for each n the probability of the first digits of X_n being in $\{1,2,3,\ldots,s-1\}$ is

$$\sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \leq X_n < s10^{-l}) = \sum_{l=-\infty}^{\infty} \int_{10^{-l}}^{s10^{-l}} \frac{h(x_n)}{(n-1)!} \left(\ln \left(\frac{1}{H(x_n)} \right) \right)^{n-1} dx_n.$$

We have, by a change of variable,

$$\sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \leq X_n < s10^{-l}) = \sum_{l=-\infty}^{\infty} \int_{\ln\left(\frac{1}{H(s10^{-l})}\right)}^{\ln\left(\frac{1}{H(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du.$$

Then, as $f(x) = h(x)$ for all $x \leq 10^{-m+1}$, we have

$$\sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \leq X_n < s10^{-l}) = \sum_{l=-\infty}^{\infty} \int_{10^{-l}}^{s10^{-l}} \frac{f(x_n)}{(n-1)!} \left(\ln \left(\frac{1}{F(x_n)} \right) \right)^{n-1} dx_n + \sum_{l=-\infty}^{m-1} \int_{\ln\left(\frac{1}{H(s10^{-l})}\right)}^{\ln\left(\frac{1}{H(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du$$
$$- \sum_{l=-\infty}^{m-1} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du.$$

Then,

$$\begin{aligned}
&\left| \sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \leq X_n < s10^{-l}) - \sum_{l=-\infty}^{\infty} \int_{10^{-l}}^{s10^{-l}} \frac{f(x_n)}{(n-1)!} \left(\ln \left(\frac{1}{F(x_n)} \right) \right)^{n-1} dx_n \right| \\
&\leq \left| \sum_{l=-\infty}^{m-1} \int_{\ln\left(\frac{1}{H(s10^{-l})}\right)}^{\ln\left(\frac{1}{H(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du \right| + \left| \sum_{l=-\infty}^{m-1} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du \right| \\
&\leq \left| \sum_{l=\delta}^{m-1} \int_{\ln\left(\frac{1}{H(s10^{-l})}\right)}^{\ln\left(\frac{1}{H(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du \right| + \left| \sum_{l=\upsilon}^{m-1} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du \right|,
\end{aligned}$$

where the last inequality follow by $H(x) = 1$ for all $x \geq 10^{-\delta+1}$ and $F(x) = 1$ for all $x \geq 10^{-\upsilon+1}$. We will show that $\sum_{l=\upsilon}^{m-1} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du \to 0$ as $n \to \infty$. Note that $\lim_{n\to\infty} \frac{u^{n-1}e^{-u}}{(n-1)!} = 0$ for all $u \in [0,\infty)$ and $\frac{u^{n-1}e^{-u}}{(n-1)!} \leq 1$. Since for each l,

$$\int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} 1 \, du = \ln \left(\frac{F(s10^{-l})}{F(10^{-l})} \right) \leq \ln \left(\frac{F(10^{-l+1})}{F(10^{-l})} \right) < \infty,$$

by Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du = \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \lim_{n\to\infty} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du = 0.$$

Then

$$\lim_{n\to\infty} \sum_{l=\upsilon}^{m-1} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du = \sum_{l=\upsilon}^{m-1} \lim_{n\to\infty} \int_{\ln\left(\frac{1}{F(s10^{-l})}\right)}^{\ln\left(\frac{1}{F(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du = 0.$$

Similarly, we have $\sum_{l=\delta}^{m-1} \int_{\ln\left(\frac{1}{H(s10^{-l})}\right)}^{\ln\left(\frac{1}{H(10^{-l})}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!} \, du \to 0$ as $n \to \infty$. Since the chain of truncated distributions of f satisfies Benford's law in the limit, so

$$\sum_{l=-\infty}^{\infty} \int_{10^{-l}}^{s10^{-l}} \frac{f(x_n)}{(n-1)!} \left(\ln \left(\frac{1}{F(x_n)} \right) \right)^{n-1} dx_n = \sum_{l=-\infty}^{\infty} \mathbb{P}\left(10^{-l} \leq Y_n < s10^{-l}\right) = \log_{10} s.$$

We have $\lim_{n\to\infty} \sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \le X_n < s10^{-l}) = \log_{10} s$ for $s \in [2, 10]$. $\square$

The following theorem and its proof are essentially the same as that in [2]. It is included here for completeness.

Theorem 3.4. *Let a be a positive constant. Let $\mathcal{U}[0,a]$ be a distribution supported on $[0,a]$ with the probability density f defined by*

$$f(x) = \frac{1}{a}, \text{ for } x \in [0,a].$$

Then the chain of truncated distributions generated by $\mathcal{U}[0,a]$ satisfies Benford's law in the limit.

Proof. Note that we mimic proof of [2]. Let X_n be a chain of truncated distribution with X_1 having density f and s an integer in $[2, 10]$. We have $f_{X_n}(x_n) = \frac{1}{a(n-1)!}\left(\ln\left(\frac{a}{x_n}\right)\right)^{n-1}$, $0 < x_n \le a$. Let $m \in \mathbb{N}$ be the minimum of $\{x \in \mathbb{Z} : 10^{-x+1} < a\}$. Then for each n, since a and $s10^{-m+1}$ lie in $(10^{-m+1}, 10^{-m+2}]$, the probability of the first digits lying between 1 and s of X_n is

$$\begin{aligned} P_n(s) = \sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \le X_n < s10^{-l}) = &\sum_{l=m}^{\infty} \int_{10^{-l}}^{s10^{-l}} \frac{1}{a(n-1)!}\left(\ln\left(\frac{a}{x_n}\right)\right)^{n-1} dx_n \\ &+ \int_{10^{-m+1}}^{\min\{s10^{-m+1},a\}} \frac{1}{a(n-1)!}\left(\ln\left(\frac{a}{x_n}\right)\right)^{n-1} dx_n. \end{aligned}$$

We have, by a change of variable,

$$\begin{aligned} P_n(s) &= \sum_{l=m}^{\infty} \int_{\ln\left(\frac{a}{s10^{-l}}\right)}^{\ln\left(\frac{a}{10^{-l}}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!}\, du + \int_{\ln\left(\frac{a}{\min\{s10^{-m+1},a\}}\right)}^{\ln\left(\frac{a}{10^{-m+1}}\right)} \frac{u^{n-1}e^{-u}}{(n-1)!}\, du \\ &= \sum_{l=m}^{\infty} \int_{\ln a + l\ln 10 - \ln s}^{\ln a + l \ln 10} \frac{u^{n-1}e^{-u}}{(n-1)!}\, du + \int_{\ln a - \ln\min\{s10^{-m+1},a\}}^{\ln a - (m-1)\ln 10} \frac{u^{n-1}e^{-u}}{(n-1)!}\, du. \end{aligned}$$

Define $g_n(u) = \begin{cases} \frac{u^{n-1}e^{-u}}{(n-1)!} & \text{if } u \ge 0; \\ 0 & \text{if } u < 0, \end{cases}$ and $M_n(s) = \int_{\ln a - \ln\min\{s10^{-m+1},a\}}^{\ln a + (m-1)\ln 10 - \ln s} \frac{u^{n-1}e^{-u}}{(n-1)!}\, du$. Then,

$$\begin{aligned} P_n(s) &= \sum_{l=m}^{\infty} \int_{\ln a + l\ln 10 - \ln s}^{\ln a + l\ln 10} g_n(u)\, du + \int_{\ln a + (m-1)\ln 10 - \ln s}^{\ln a + (m-1)\ln 10} g_n(u)\, du + M_n(s) \\ &= \sum_{l=-\infty}^{\infty} \int_{\ln a + l\ln 10 - \ln s}^{\ln a + l\ln 10} g_n(u)\, du + M_n(s) \qquad \left(\text{because } 10^{-l} \ge a \text{ for all } l < m-1\right) \\ &= \sum_{l=-\infty}^{\infty} \int_{\ln a - \ln s}^{\ln a} g_n(u + l\ln 10)\, du + M_n(s) \\ &= \sum_{l=-\infty}^{\infty} \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} (\ln 10)\, g_n(u\ln 10 + l\ln 10)\, du + M_n(s) \\ &= (\ln 10) \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} \left(\sum_{l=-\infty}^{\infty} g_n(u\ln 10 + l\ln 10) \right) du + M_n(s). \end{aligned}$$

By the Poisson Summation Formula, we have

$$\sum_{l=-\infty}^{\infty} g_n(u\ln 10 + l\ln 10) = \sum_{l=-\infty}^{\infty} \left(\frac{1}{\ln 10}\right) e^{2\pi i l u} \widehat{g_n}\left(\frac{l}{\ln 10}\right).$$

Note that the Fourier transform of the Gamma density function g_n with the parameters n and 1 is $\widehat{g_n}(t) = \varphi(-2\pi it) = (1+2\pi it)^{-n}$, where φ is the characteristic function of Gamma distribution with the parameters n and 1. See [5]. Then,

$$\begin{aligned}
P_n(s) &= \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} \left(\sum_{l=-\infty}^{\infty} e^{2\pi ilu} \widehat{g_n}\left(\frac{l}{\ln 10}\right) \right) du + M_n(s) \\
&= \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} \left(\sum_{l=-\infty}^{\infty} e^{2\pi ilu} \left(1 + \frac{2\pi il}{\ln 10}\right)^{-n} \right) du + M_n(s) \\
&= \log_{10} s + \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} \left(\sum_{l\neq 0} e^{2\pi ilu} \left(1 + \frac{2\pi il}{\ln 10}\right)^{-n} \right) du + M_n(s).
\end{aligned}$$

Hence,

$$\begin{aligned}
|P_n(s) - \log_{10} s| &= \left| \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} \sum_{l\neq 0} e^{2\pi ilu} \left(1 + \frac{2\pi il}{\ln 10}\right)^{-n} du + M_n(s) \right| \\
&\leq \left| \int_{\log_{10} a - \log_{10} s}^{\log_{10} a} \sum_{l\neq 0} e^{2\pi ilu} \left(1 + \frac{2\pi il}{\ln 10}\right)^{-n} du \right| + |M_n(s)| \\
&\leq \log_{10} s \sum_{l\neq 0} \left| \left(1 + \frac{2\pi il}{\ln 10}\right)^{-n} \right| + |M_n(s)| \\
&= 2\log_{10} s \sum_{l=1}^{\infty} \left(\sqrt{1 + \left(\frac{2\pi l}{\ln 10}\right)^2} \right)^{-n} + |M_n(s)|.
\end{aligned}$$

We will show that $|M_n(s)| \to 0$ as $n \to \infty$. Note that $\lim_{n\to\infty} \frac{u^{n-1}e^{-u}}{(n-1)!} = 0$ for all $u \in [0, \infty)$ and $\frac{u^{n-1}e^{-u}}{(n-1)!} \leq 1$. Since

$$\int_{\ln a - \ln \min\{s10^{-m+1}, a\}}^{\ln a + (m-1)\ln 10 - \ln s} 1\, du = (m-1)\ln 10 - \ln s + \ln \min\left\{s10^{-m+1}, a\right\} < \infty,$$

by DCT, we have

$$\lim_{n\to\infty} \left(\int_{\ln a - \ln \min\{s10^{-m+1}, a\}}^{\ln a + (m-1)\ln 10 - \ln s} \frac{u^{n-1}e^{-u}}{(n-1)!} du \right) = \int_{\ln a - \ln \min\{s10^{-m+1}, a\}}^{\ln a + (m-1)\ln 10 - \ln s} \lim_{n\to\infty} \frac{u^{n-1}e^{-u}}{(n-1)!} du = 0.$$

We will show that $\sum_{l=1}^{\infty} \left(\sqrt{1 + \left(\frac{2\pi l}{\ln 10}\right)^2} \right)^{-n} \to 0$ as $n \to \infty$. Since

$$\sum_{l=1}^{\infty} \left(\sqrt{1 + \left(\frac{2\pi l}{\ln 10}\right)^2} \right)^{-n} \leq \sum_{l=1}^{\infty} \left(\frac{2\pi l}{\ln 10}\right)^{-n} = \left(\frac{\ln 10}{2\pi}\right)^{n} \sum_{l=1}^{\infty} \frac{1}{l^n}$$

and $\sum_{l=1}^{\infty} \frac{1}{l^n}$ converges for all $n > 1$ and $\left(\frac{\ln 10}{2\pi}\right)^{n} \to 0$ as $n \to \infty$.

So $\lim_{n\to\infty} \sum_{l=1}^{\infty} \left(\sqrt{1 + \left(\frac{2\pi l}{\ln 10}\right)^2} \right)^{-n} = 0$. Thus, $\lim_{n\to\infty} \sum_{l=-\infty}^{\infty} \mathbb{P}(10^{-l} \leq X_n < s10^{-l}) = \log_{10} s$. Then the chain of truncated distributions generated by $\mathcal{U}[0, a]$ satisfies Benford's law in the limit. □

Combining Theorems 3.3 and 3.4, we have:

Corollary 3.5. *Let a be a positive constant. Let $\mathcal{D}[0,k]$, $k > 0$ be a distribution with the probability density f defined by*

$$f(x) = \begin{cases} a, & \text{if } x \in [0,\varepsilon), \\ h(x), & \text{if } x \in [\varepsilon, k]. \end{cases}$$

for some integrable function h and $\varepsilon \in (0,k]$. Then the chain of truncated distributions generated by $\mathcal{D}$ satisfies Benford's law in the limit.

Acknowledgment. The authors are grateful to the referees for their careful reading of the manuscript and their useful comments.

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