New exact solutions for the time fractional clannish random walker’s parabolic equation by the improved \( \tan(\phi(\xi)/2) \)-expansion method

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Abstract

In this paper, the fractional order derivative and the improved \( \tan(\phi(\xi)/2) \) - expansion method are proposed to construct the abundant exact solutions of nonlinear time fractional partial differential equations. For illustrating the validity of the improved \( \tan(\phi(\xi)/2) \)-expansion method which is direct, efficient and powerful, the method applied to solve exact solutions the time fractional Clannish Random Walkers Parabolic equation that applies to determine the density of species. As the result, some new general exact solutions expressed in various forms including the trigonometric function solutions, hyperbolic function solutions, and a rational function solution which are found by the aid of mathematical software Maple program. This method is not only an powerful mathematical tool for generating more solutions of nonlinear time fractional partial differential equations but also can applied to nonlinear space-time fractional partial differential equations.

Keywords: Exact solutions, The time fractional differential equations, Local fractional derivative, The improved \( \tan(\phi(\xi)/2) \) - expansion method.


1 Introduction

In recent years, fractional order partial differential equations (FPDEs) play an important role in various scientific and engineering fields and has had increasing application in many fields, including physics, biology, finance, fractional dynamics [1–3], signal processing, electromagnetism, systems identification, electrochemistry, control theory, fluid mechanics [4] and engineering [5–7]. The rapid development of efficient techniques for solving fractional order partial differential equations (FPDEs) have been of great help in tackling real world problems of a highly complex nature. Therefore, researchers have been interested in studying the fractional calculus and finding accurate and efficient methods for seeking exact solutions of fractional order partial differential equations (FPDEs).

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An important area of research is finding exact solutions of both fractional order ordinary differential equations (FODEs) and fractional order partial differential equations (FPDEs). In the past few decades, many powerful and efficient methods have been developed to obtain exact solutions of FODEs and FPDEs. These methods include the homotopy perturbation method [8,9], the simplest equation method [10], the Ansatz method [11–14], the Riccati equation method [15,16], the tan(ϕ(ξ)/2)-expansion method [19,20], the Kudryashov method [21,22], the modified trial equation method [23,24], the first integral method [25,26], the fractional G'/G-expansion method [27–30], the fractional expo-expansion method [31–33], the fractional sub-equation method [34–36], the fractional functional variable method [37], the fractional modified trial equation method [23,38], etc.

The improved tan (ϕ(ξ)/2)-expansion method which can solve exact solutions of NLPDEs and FPDEs was introduced by Manaian et al. [39,40]. The method is a simple and a powerful mathematical tool for constructing exact solutions that obtain nineteen different types exact solutions of a range of NLPDEs and FPDEs.

In this paper, we further construct more exact solutions for the time fractional Clan-nish Random Walkers Parabolic equation (CRWP) by introducing the improved tan (ϕ(ξ)/2)-expansion method which provides different forms of nineteen solutions. The CRWP equation [41] is a model that can determine the behaviour of two species A and B of random walkers who execute a concurrent one-dimensional random walk characterized by an intensification of the clannishness of the members of one species as the density of the other increases. The density of specie A at point x at time t, u(x,t), can be expressed by the time fractional Clannish Random Walkers Parabolic equation as:

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial u(x,t)}{\partial x} + 2u(x,t) \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad x \in R, \quad (1.1) \]

where \( \alpha \) is a parameter describing the order of the fractional time derivative and \( 0 < \alpha \leq 1 \).

2 Definitions of the modified Reimann-Liouville derivative

In this section, we present the definitions and some important properties of the modified Riemann-Liouville derivative of order \( \alpha \) as: [42–47].

\[ D_\alpha^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0))d\xi, & 0 < \alpha < 1 \\ (f^n(t))^{(\alpha-n)}, & n \leq \alpha < n+1, \ n \geq 1. \end{cases} \]

where \( f : R \rightarrow R \), denotes a continuous (but not necessarily first-order-differentiable) function. Some used formulas are given by

1. \( D_\alpha^\alpha t = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha} \), \n
2. \( D_\alpha^\alpha (cf(t)) = cD_\alpha^\alpha f(t), \quad \text{c = constant}, \)
3. \( D_\alpha^\alpha c = 0, \quad \text{c = constant}, \)
4. \( D_\alpha^\alpha [af(t) + bg(t)] = aD_\alpha^\alpha f(t) + bD_\alpha^\alpha g(t), \) where \( a \) and \( b \) are constants,
5. \( D_\alpha^\alpha f(t)g(t) = g(t)D_\alpha^\alpha f(t) + f(t)D_\alpha^\alpha g(t), \)
6. \( D_\alpha^\alpha f(g(t)) = f'(g(t))D_\alpha^\alpha g(t) = D_\alpha^\alpha f(g(t))(g'(t))^\alpha. \)

3 Description of the improved \( \tan(\phi(\xi)/2) \)-expansion method

Suppose that a general fractional partial differential equation in the independent variables of point \( x \) at time \( t \) is given by

\[ P(u, u_x, u_{xx}, D_\alpha^\alpha u, D_x u, D_x^2 u, \ldots) = 0, \quad 0 < \alpha \leq 1, \quad (3.1) \]

where \( p \) is a polynomial of \( u \) and \( D_\alpha^\alpha u \) are fractional order \( \alpha \) partial derivatives, the function \( (u) \) denotes a solution. Next, we illustrate the description of the improved \( \tan(\phi(\xi)/2) \)-expansion
method for seeking exact solutions of a fractional partial differential equation in the following steps.

Step 1: Suppose \( u(x,t) = U(\xi) \), and a nonlinear fractional complex transformation

\[
\xi = kx - \frac{rt^\alpha}{\Gamma(1 + \alpha)}.
\]

(3.2)

Then by the first equality of the property 6, the equation (3.1) can be reduced to the following nonlinear ordinary differential equation with respect to the variable \( \xi \): By using the chain rule

\[
D^\alpha_t u = \frac{dU}{d\xi} D^\alpha_t \xi
\]

\[
= U' D^\alpha_t \left( kx - \frac{rt^\alpha}{\Gamma(1 + \alpha)} \right) = U' \left( -\frac{r}{\Gamma(1 + \alpha)} \frac{1}{\Gamma(1)} \right) = -r U'
\]

(3.3)

\[
D_x u = \frac{dU}{d\xi} D_x \xi
\]

\[
= U' D_x \left( kx - \frac{rt^\alpha}{\Gamma(1 + \alpha)} \right) = k U'
\]

(3.4)

Substituting \( u(x,t) = U(\xi) \) with formulas in Eqs. (3.3) and (3.4) into (3.1), we can rewrite Eq.(3.1) in the following nonlinear ODE:

\[
Q(U, U', U'', U''', ...) = 0,
\]

(3.5)

where \( ' \) denotes the derivatization with respect to \( \xi \). We should integrate Eq.(3.5) term by term as soon as possible.

Step 2: Suppose a solution of Eq.(3.5) can be expressed as follows:

\[
U(\xi) = \sum_{k=0}^{m} a_k \left[ p + \tan \left( \frac{\phi(\xi)}{2} \right) \right]^k + \sum_{k=1}^{m} b_k \left[ p + \tan \left( \frac{\phi(\xi)}{2} \right) \right]^{-k},
\]

(3.6)

where \( p, a_k (0 \leq k \leq m) \) and \( b_k (1 \leq k \leq m) \) are constants to be determined, such that \( a_m \neq 0 \), \( b_m \neq 0 \) and the function \( \phi(\xi) \) satisfies the following ordinary differential equation:

\[
\phi' (\xi) = a \sin \phi(\xi) + b \cos \phi(\xi) + c,
\]

(3.7)

where \( a, b \) and \( c \) are constants. Moreover the positive integer \( m \) can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms of appearing in Eq.(3.5) where the degree of the expressions will be determined by the following formulas:

\[
D[U(\xi)] = m; \quad D \left[ \frac{d^p U(\xi)}{d\xi^p} \right] = m + p, \quad D \left[ U^p \left( \frac{d^q U(\xi)}{d\xi^q} \right)^s \right] = mp + s(m + q).
\]

(3.8)

We will consider the following special solutions of equation Eq.(3.7):

Family 1: When \( a^2 + b^2 - c^2 < 0 \) and \( b - c \neq 0 \), then

\[
\phi(\xi) = 2 \text{arctan} \left[ \frac{a}{b - c} - \sqrt{\frac{c^2 - b^2 - a^2}{b - c}} \tan \left( \frac{\sqrt{c^2 - b^2 - a^2}}{2} (\xi + C) \right) \right].
\]

(3.9)

Family 2: When \( a^2 + b^2 - c^2 > 0 \) and \( b - c \neq 0 \), then

\[
\phi(\xi) = 2 \text{arctan} \left[ \frac{a}{b - c} + \sqrt{\frac{b^2 + a^2 - c^2}{b - c}} \tanh \left( \frac{\sqrt{b^2 + a^2 - c^2}}{2} (\xi + C) \right) \right].
\]

(3.10)

Family 3: When \( a^2 + b^2 - c^2 > 0, b \neq 0 \) and \( c = 0 \), then

\[
\phi(\xi) = 2 \text{arctan} \left[ \frac{a}{b} + \sqrt{\frac{b^2 + a^2}{b}} \tanh \left( \frac{\sqrt{b^2 + a^2}}{2} (\xi + C) \right) \right].
\]

(3.11)
Family 4: When \(a^2 + b^2 - c^2 < 0, b \neq 0\) and \(c = 0\), then
\[
\phi(\xi) = 2 \arctan \left[ -\frac{a}{c} + \sqrt{\frac{c^2 - a^2}{c}} \tan \left( \frac{\sqrt{c^2 - a^2}}{2} (\xi + C) \right) \right].
\] (3.12)

Family 5: When \(a^2 + b^2 - c^2 < 0, b - c \neq 0\) and \(a = 0\), then
\[
\phi(\xi) = 2 \arctan \left[ \frac{b + c}{b - c} \tan \left( \frac{\sqrt{b^2 - c^2}}{2} (\xi + C) \right) \right].
\] (3.13)

Family 6: When \(a = 0\) and \(c = 0\), then
\[
\phi(\xi) = \arctan \left[ \frac{e^{2b(\xi+c)} - 1}{e^{2b(\xi+c)} + 1 \cdot e^{2b(\xi+c)} + 1} \right].
\] (3.14)

Family 7: When \(b = 0\) and \(c = 0\), then
\[
\phi(\xi) = \arctan \left[ \frac{2e^{a(\xi+c)} - 1}{e^{2a(\xi+c)} + 1 \cdot e^{2a(\xi+c)} + 1} \right].
\] (3.15)

Family 8: When \(a^2 + b^2 = c^2\), then
\[
\phi(\xi) = -2 \arctan \left[ \frac{b + c(a(\xi + c) + 2)}{a^2(\xi + 2)} \right].
\] (3.16)

Family 9: When \(a = b = c = ka\), then
\[
\phi(\xi) = 2 \arctan \left[ e^{ka(\xi+c)} - 1 \right].
\] (3.17)

Family 10: When \(a = c = ka\) and \(b = -ka\), then
\[
\phi(\xi) = -2 \arctan \left[ \frac{e^{ka(\xi+c)}}{-1 + e^{ka(\xi+c)}} \right].
\] (3.18)

Family 11: When \(c = a\), then
\[
\phi(\xi) = -2 \arctan \left[ \frac{(a + b)e^{b(\xi+c)} - 1}{(a - b)e^{b(\xi+c)} - 1} \right].
\] (3.19)

Family 12: When \(a = c\), then
\[
\phi(\xi) = 2 \arctan \left[ \frac{(b + c)e^{b(\xi+c)} + 1}{(b - c)e^{b(\xi+c)} - 1} \right].
\] (3.20)

Family 13: When \(c = -a\), then
\[
\phi(\xi) = 2 \arctan \left[ \frac{e^{b(\xi+c)} + b - a}{e^{b(\xi+c)} - b - a} \right].
\] (3.21)

Family 14: When \(b = -c\), then
\[
\phi(\xi) = 2 \arctan \left[ \frac{ae^{a(\xi+c)} + 1}{1 - e^{a(\xi+c)}} \right].
\] (3.22)

Family 15: When \(b = 0\) and \(a = c\), then
\[
\phi(\xi) = -2 \arctan \left[ \frac{c(\xi + C) + 2}{c(\xi + C)} \right].
\] (3.23)

Family 16: When \(a = 0\) and \(b = c\), then
\[
\phi(\xi) = 2 \arctan \left[ c(\xi + C) \right].
\] (3.24)
Family 17: When $a = 0$ and $b = -c$, then
\[ \phi(\xi) = -2 \arctan \left( \frac{1}{c(\xi + C)} \right). \] (3.25)

Family 18: When $a = 0$ and $b = 0$, then
\[ \phi(\xi) = c(\xi + C). \] (3.26)

Family 19: When $a = 0$ and $b = 0$, then
\[ \phi(\xi) = 2 \arctan \left( \frac{e^{a(\xi+C)} - c}{a} \right). \] (3.27)

Step 3: Substituting (3.6) into Eq.(3.5) with the value of $m$ that obtained in Step 1. Collecting the coefficients of $\tan \left( \frac{\phi(\xi)/2}{k} \right)$, $\cot \left( \frac{\phi(\xi)/2}{k} \right)$ ($k = 0, 1, 2, ..., m$), then setting each coefficient to zero, we can get a set of over-determined the system of equations for $a_0, a_k, b_k (k = 1, 2, ..., m), a, b, c$ and $p$ with the aid of symbolic computation using Maple.

Step 4: Solving the system of algebraic equations in step 2, then all solutions of Eq.(3.1) are obtained by substituting the values of $a_0, a_1, b_1, ..., a_m, b_m, a, b, c$ and $p$ along with the solutions Eq.(3.9)-(3.27) from all families into (3.6).

4 Exact solution of the time fractional CRWP equation

In this section, we apply the improved $\tan \left( \frac{\phi(\xi)/2}{k} \right)$-expansion method to the time fractional Clannish Random Walker’s Parabolic equation (CRWP) as follows:
\[ \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial u(x, t)}{\partial x} + 2u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{\partial^2 u(x, t)}{\partial x^2} = 0, \quad x \in R, \] (4.1)
where $\alpha$ is the order of the fractional time derivative and $0 < \alpha \leq 1$ and $u$ denoted a density of a species. We take the nonlinear fractional transformation
\[ u(x, t) = U(\xi), \quad \xi = kx - \frac{rt^\alpha}{\Gamma(1 + \alpha)}; \] (4.2)
where $r$ and $k$ is a non-zero arbitrary constant and all fractional order derivatives as:
\[ \frac{\partial^\alpha u}{\partial t^\alpha} = U' \frac{\partial^\alpha}{\partial t^\alpha} \left[ kx - \frac{rt^\alpha}{\Gamma(1 + \alpha)} \right] = U' \left[ -\frac{r}{\Gamma(1 + \alpha)} \cdot \frac{\Gamma(1 + \alpha)}{\Gamma(1)} \cdot t^{\alpha-a} \right] = -rU', \]
\[ \frac{\partial u}{\partial x} = ku', \quad \frac{\partial^2 u}{\partial x^2} = k^2 U''. \] (4.3)
Substituting into Eq.(4.1), the Eq.(4.1) can be reduced to the integrable nonlinear differential equation
\[ (k - r)U' + 2kU'' + k^2 U''' = 0, \] (4.4)
Integrating both side of the equation, we obtain
\[ (k - r)U + kU^2 + k^2 U' + \xi_0 = 0, \] (4.5)
where $\xi_0$ is an integration constant.
Now, suppose that the solution of Eq.(4.5) can be expressed by
\[ U(\xi) = \sum_{k=0}^{m} a_k \left[ p + \tan \left( \frac{\phi(\xi)}{2} \right) \right]^k + \sum_{k=1}^{m} b_k \left[ p + \tan \left( \frac{\phi(\xi)}{2} \right) \right]^{-k}, \] (4.6)
where $\phi(\xi)$ satisfies (3.7). The value of the constant $m$ is determined by considering the degree of $U'$ in Eq.(4.5) expressed by $U' : D \left( \frac{d^p U(\xi)}{d\xi^p} \right)$ is $m + 1$, while the degree of $U^2$ expressed by $U^2 : D \left[ U^p \left( \frac{d^p U(\xi)}{d\xi^p} \right) \right]$ is $2m$, by balancing the order between $U'$ and $U^2$ in Eq.(4.5), one can obtain $m = 1$. So one has

$$U(\xi) = a_0 + a_1 \left( p + \tan \left( \frac{\phi(\xi)}{2} \right) \right) + \frac{b_1}{p + \tan \left( \frac{\phi(\xi)}{2} \right)}, \quad (4.7)$$

$$U^2(\xi) = a_0^2 + 2pa_0a_1 + 2a_0a_1 \tan \left( \frac{\phi(\xi)}{2} \right) + \frac{2a_0b_1}{p + \tan \left( \frac{\phi(\xi)}{2} \right)} + a_1^2 p^2 + 2pa_1 \tan \left( \frac{\phi(\xi)}{2} \right) + \frac{2pa_1b_1}{p + \tan \left( \frac{\phi(\xi)}{2} \right)} + a_1^2 \tan^2 \left( \frac{\phi(\xi)}{2} \right) + \frac{b_1^2}{(p + \tan \left( \frac{\phi(\xi)}{2} \right))^2}, \quad (4.8)$$

and the first derivative as:

$$U'(\xi) = \frac{1}{2} a_1 \left( 1 + \tan^2 \left( \frac{\phi(\xi)}{2} \right) \right) \phi'(\xi) - \frac{b_1 \left( 1 + \tan^2 \left( \frac{\phi(\xi)}{2} \right) \right) \phi'(\xi)}{(p + \tan \left( \frac{\phi(\xi)}{2} \right))^2}. \quad (4.9)$$

From Eq.(3.7), we have $\phi'(\xi) = a \sin \phi(\xi) + b \cos \phi(\xi) + c$ by trigonometric identities as:

$$\sin \phi(\xi) = \frac{2 \tan(\phi(\xi)/2)}{1 + \tan^2(\phi(\xi)/2)}, \quad \text{and} \quad \cos \phi(\xi) = \frac{1 - \tan^2(\phi(\xi)/2)}{1 + \tan^2(\phi(\xi)/2)}. \quad (4.10)$$

Substitute them into Eq.(3.7), we obtain

$$\phi'(\xi) = \frac{2a \tan \left( \frac{\phi(\xi)}{2} \right)}{1 + \tan^2 \left( \frac{\phi(\xi)}{2} \right)} + \frac{b \left( 1 - \tan^2 \left( \frac{\phi(\xi)}{2} \right) \right)}{1 + \tan^2 \left( \frac{\phi(\xi)}{2} \right)} + c. \quad (4.11)$$

Let $\tan \left( \frac{\phi(\xi)}{2} \right) = Y$, it obtains

$$\phi'(\xi) = \frac{2aY}{Y^2 + 1} + \frac{b(1 - Y^2)}{Y^2 + 1} + c. \quad (4.12)$$

$$U(\xi) = a_0 + a_1 \left( p + Y \right) + \frac{b_1}{p + Y}, \quad (4.13)$$

$$U^2(\xi) = a_0^2 + 2pa_0a_1 + 2a_0a_1 Y^2 + 2pa_1 Y + a_1^2 p^2 + \frac{2a_0b_1}{p + Y} + \frac{2pa_1b_1}{p + Y} + \frac{b_1^2}{(p + Y)^2}, \quad (4.14)$$

$$U'(\xi) = \frac{1}{2} a_1 \left( Y^2 + 1 \right) \phi'(\xi) - \frac{b_1 \left( Y^2 + 1 \right) \phi'(\xi)}{2(p + Y)^2}. \quad (4.15)$$
Substituting Eq. (4.12) into Eq. (4.15), we have

\[
U'(\xi) = \frac{1}{2}a_1(Y^2 + 1) \left( \frac{2aY}{Y^2 + 1} + b(1 - Y^2) + c \right) - \frac{b_1(Y^2 + 1) \left( \frac{2aY}{Y^2 + 1} + \frac{b(1 - Y^2)}{Y^2 + 1} + c \right)}{2(p + Y^2)^2} = \frac{\left(a_1Y^2 + 2pa_1Y + p^2a_1 - b_1\right)\left(bY^2 - cY^2 - 2aY - b - c\right)}{2(p + Y^2)^2}. \quad (4.16)
\]

Substituting \( U(\xi), U^2(\xi) \) and \( U'(\xi) \) into Eq.(4.5) and collecting all the terms with same order of \( Y^j, j = 0, 1, 2, 3, 4 \) and equating each coefficient to zero, it obtains the system of algebraic equations as

\[
\begin{align*}
Y^0 & : 2a_1^2k^2p^4 + 4a_0a_1k^3p + a_1bk^2p^2 + a_1ck^2p^2 + 2a_0^2k^2p^2 + 4a_1b_1k^2p^2 + 2a_1k^3p^3 \\
& \quad - 2a_1rp^3 + 4a_0b_1kp + 2a_0k^2p - 2a_0rp^2 - bb_1k^2 - b_1ck^2 + 2b_1k + 2b_1kp - 2b_1rp + 2p^2\xi_0 = 0, \\
Y^1 & : 2a_1k^3p^2 + 8a_1^2k^3p + 12a_0a_1k^2p + 2a_1bk^2p + 2a_1ck^2p - 2ab_1k^2 + 4a_0k^2p + 8a_1b_1kp + 6a_1k^2p - 6a_1p^2r + 4a_0b_1k + 4a_0k^2p - 4a_0pr + 2b_1k - 2b_1r + 4p\xi_0 = 0, \\
Y^2 & : -a_1bk^2p^2 + a_1ck^2p^2 + 4a_1k^2p^2 + 12a_0a_1k^2p + 12a_0k^2p + a_1bk^2 + a_1ck^2 \\
& \quad + bb_1k^2 - b_1ck^2 + 2a_0k^2 + 4a_1b_1k + 6a_1k^2p - 6a_1pr + 2a_0k - 2a_0r + 2\xi_0 = 0, \\
Y^3 & : -2a_1bk^2p^2 + 2a_1ck^2p^2 + 2a_1k^2p^2 + 8a_1^2k^2p + 4a_0a_1k + 2a_1k - 2a_1r = 0, \\
Y^4 & : -a_1bk^2 + a_1ck^2 + 2a_1^2k = 0.
\end{align*}
\]

Solving the system of algebraic equations with the aid the symbolic mathematical software Maple yields two cases of the following values of the coefficients \( a_0, a_1, b_1, k, r \) and \( \xi_0 \).

**case 1:** \( a_0 = a_0, a_1 = 0, b_1 = -\frac{1}{2}k(bp^2 - cp^2 + 2ap - b - c), k = k, \)

\[
r = -k(bkp - ckp + ak - 2a_0 - 1),
\]

\[
\xi_0 = \frac{1}{4}k(-b^2k^2p^2 + 2bck^2p^2 - c^2k^2p^2 - 2abk^2p + 2ack^2p + 4a_0bp - 4a_0ckp + b^2k^2 - c^2k^2 + 4aaok - 4a_0^2).
\]

**case 2:** \( a_0 = a_0, a_1 = \frac{1}{2}k(b - c), b_1 = 0, k = k, \)

\[
r = k(bkp - ckp + ak + 2a_0 + 1),
\]

\[
\xi_0 = -\frac{1}{4}k(-b^2k^2p^2 + 2bck^2p^2 - c^2k^2p^2 - 2abk^2p + 2ack^2p + 4a_0bp - 4a_0ckp + b^2k^2 - c^2k^2 + 4aaok - 4a_0^2).
\]

Substituting these results into the equation (4.7), and combining with all solutions (3.9)-(3.27) of Eq.(3.7), one can obtain abundant exact solutions including trigonometric function solutions, hyperbolic function solutions, exponential function solutions and rational function solutions.

From **Case 1** and the transformation \( \xi = kx - \frac{\tau^\alpha}{\Gamma(1+\alpha)} \), the improved tan(\( \phi(\xi)/2 \))-expansion method provides the exact solutions of the time fractional Clannish Random Walker’s Parabolic
equation in Eq.(4.1) as:

\[
\begin{align*}
u_{1,1}(x, t) &= a_0 + \frac{-\frac{1}{2}k(bp^2 - cp^2 + 2ap - b - c)}{p + \frac{a}{b-c} - \sqrt{c^2-b^2-a^2} \tan \left( \frac{\sqrt{c^2-b^2-a^2}}{2} (\xi + C) \right)}, \\
u_{1,2}(x, t) &= a_0 + \frac{-\frac{1}{2}(bp^2 - cp^2 + 2ap - b - c)}{p + \frac{a}{b-c} + \sqrt{a^2+b^2-c^2} \tan \left( \frac{\sqrt{a^2+b^2-c^2}}{2} (\xi + C) \right)}, \\
u_{1,3}(x, t) &= a_0 + \frac{-\frac{1}{2}k(bp^2 - cp^2 + 2ap - b - c)}{p + \frac{a}{b-c} + \sqrt{a^2+b^2-c^2} \tan \left( \frac{\sqrt{a^2+b^2-c^2}}{2} (\xi + C) \right)}, \\
u_{1,4}(x, t) &= a_0 + \frac{-\frac{1}{2}k(-bp^2 + 2ap - c)}{p - \frac{a}{b-c} + \sqrt{a^2+b^2-c^2} \tan \left( \frac{\sqrt{a^2+b^2-c^2}}{2} (\xi + C) \right)}, \\
u_{1,5}(x, t) &= a_0 + \frac{\frac{1}{2}k(b^4p^2 - bp^2 - 2ap + \sqrt{a^2+b^2}}{p - \frac{(b+\sqrt{a^2+b^2})}{(a+\xi+C+2)}} a^2(\xi+C). \\
u_{1,6}(x, t) &= a_0 + \frac{-dpk^2 + dk^2}{p + e^{k(dk+\xi)} - 1}, \\
u_{1,7}(x, t) &= a_0 + \frac{dk^2p^2 - dk^2p}{p - \frac{e^{k(dk+\xi)}}{e^{k(dk+\xi)}-1}}, \\
u_{1,8}(x, t) &= a_0 + \frac{-\frac{1}{2}k(p-1)(bp - cp + b + c)}{p - \frac{(b+c)e^{k(\xi+C)+1}}{(c-b)e^{k(\xi+C)-1}}}, \\
u_{1,9}(x, t) &= a_0 + \frac{-\frac{1}{2}k(p-1)(bp - cp + b + c)}{p + \frac{(b+c)e^{k(\xi+C)-1}}{(c-b)e^{k(\xi+C)-1}}}, \\
u_{1,10}(x, t) &= a_0 + \frac{-\frac{1}{2}k(p+1)(bp - cp - b - c)}{p + \frac{e^{k(\xi+C)+b+c}}{e^{k(\xi+C)-b+c}}}, \\
u_{1,11}(x, t) &= a_0 + \frac{ckp^2 - akp}{p + \frac{ck}{1 - e^{ak(\xi+C)}}}, \\
u_{1,12}(x, t) &= a_0 + \frac{\frac{1}{2}ck(p-1)^2}{p - \frac{c}{c^{\xi+C+2}}}, \\
u_{1,13}(x, t) &= a_0 + \frac{ck}{p + c(\xi+C)}, \\
u_{1,14}(x, t) &= a_0 + \frac{kcp^2}{p - \frac{c}{c^{\xi+C}}}, \\
u_{1,15}(x, t) &= a_0 + \frac{\frac{1}{2}ck(p^2 + 1)}{p + \tan \left( \frac{1}{2}c\xi + \frac{1}{2}C \right)}, \\
u_{1,16}(x, t) &= a_0 + \frac{-akp + ck}{p + \frac{e^{ak(\xi+C)-1}}{a}},
\end{align*}
\]

where \(\xi = kx - \frac{r^\alpha}{1+\alpha}\).

From case 2, we have the values of the parameters

\[
\begin{align*}
a_0 &= a_0, \quad a_1 = \frac{1}{2}k(b-c), \quad b_1 = 0, \quad k = k, \quad r = k(bkp - ckp + ak + 2a_0 + 1), \\
\xi_0 &= \frac{-\frac{1}{4}k(-b^2k^2p^2 + 2bck^2p^2 - c^2k^2p^2 - 2abk^2p + 2ack^2p + 4a_0bkp - 4a_0ckp + b^2k^2 - c^2k^2 + 4aa_0k - 4a_0^2)}{p + \frac{e^{ak(\xi+C)-1}}{a}},
\end{align*}
\]

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then the improved tan(\(\phi(\xi)/2\))-expansion method can obtain the exact solutions of the time fractional Clannish Random Walker’s Parabolic equation in Eq. (4.1) as following:

\[
\begin{align*}
    u_{2,1}(x, t) &= a_0 + \frac{1}{2} k(b - c) \left( p + \frac{a}{b - c} - \frac{\sqrt{c^2 - b^2 - a^2}}{b - c} \tan \left( \frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C) \right) \right), \\
    u_{2,2}(x, t) &= a_0 + \frac{1}{2} k(b - c) \left( p + \frac{a}{b - c} + \frac{\sqrt{a^2 + b^2 - c^2}}{b - c} \tanh \left( \frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C) \right) \right), \\
    u_{2,3}(x, t) &= a_0 + \frac{1}{2} b k \left( p + \frac{a + \sqrt{a^2 + b^2}}{b} \tanh \left( \frac{\sqrt{a^2 + b^2}}{2}(\xi + C) \right) \right), \\
    u_{2,4}(x, t) &= a_0 - \frac{1}{2} c k \left( p - \frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{c} \tan \left( \frac{\sqrt{c^2 - a^2}}{2}(\xi + C) \right) \right), \\
    u_{2,5}(x, t) &= a_0 - \frac{1}{2} k(\sqrt{a^2 + b^2} - b) \left( p - \frac{b + \sqrt{a^2 + b^2}}{a^2}(\xi + C) \right), \\
    u_{2,6}(x, t) &= a_0 - d k^2 \left( p - \frac{e^{kd(kd + \xi)}}{e^{kd(kd + \xi)} - 1} \right), \\
    u_{2,7}(x, t) &= a_0 + \frac{1}{2} k(b - c) \left( p - \frac{(b + c)e^{b(\xi + C)} - 1}{(c - b)e^{b(\xi + C)} - 1} \right), \\
    u_{2,8}(x, t) &= a_0 + \frac{1}{2} k(b - c) \left( p + \frac{(b + c)e^{b(\xi + C)} + 1}{(b - c)e^{b(\xi + C)} - 1} \right), \\
    u_{2,9}(x, t) &= a_0 + \frac{1}{2} k(b - c) \left( p + \frac{e^{b(\xi + C)} + b + c}{e^{b(\xi + C)} - b + c} \right), \\
    u_{2,10}(x, t) &= a_0 - c k \left( p + \frac{ae^{a(\xi + C)}}{1 - ce^{a(\xi + C)}} \right), \\
    u_{2,11}(x, t) &= a_0 - \frac{1}{2} c k \left( p - \frac{c(\xi + C) + 2}{c(\xi + C)} \right), \\
    u_{2,12}(x, t) &= a_0 - c k \left( p - \frac{1}{c(\xi + C)} \right), \\
    u_{2,13}(x, t) &= a_0 - \frac{1}{2} c k \left( p + \tan \left( \frac{1}{2} c \xi + \frac{1}{2} C \right) \right),
\end{align*}
\]

(4.23)

where \( \xi = kx - \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \).

5 Graphical illustrations of the solutions

The graphs of some exact solutions of the time fractional Clannish Random Walker’s Parabolic equation in Eq. (4.1) are shown in Fig.(1)-(2) with the help of the improved tan(\(\phi(\xi)/2\))-expansion method, when given three parameters \( a = 0, b = 2, \) and \( c = 2 \), the coefficients \( a_0 = 2, a_1 = 0, b_1 = 2 \). The Figure (1) shows the solutions of \( u_{1,13}(x, t) \) from Eq. (4.22) in a dimensional figure with \( 0 \leq x \leq 100, 0 \leq t \leq 100 \) and three different values for fractional order \( \alpha = 0.29, 0.52, 0.73 \).
The Figure (2) shows the solutions of $u_{2,13}(x,t)$ from Eq. (4.23) in a three dimensional figure with the parameters $a = 0$, $b = 0$, and $c = 2$, the coefficients $a_0 = 2, a_1 = -1, b_1 = 0$, and letting three different values for fractional order $\alpha = 0.33, 0.59, 0.93$.

6 Conclusions

With helping of the fractional order derivative and the modified Reimann-Liouville derivative, the improved tan($\phi(\xi)/2$)-expansion method can obtain more exact solutions of the time fractional Clannish Random Walker’s Parabolic equation. This method is based on homogeneous balancing principle and a variety of cases of solution by solving the system of algebraic equations that are collected from setting coefficients to be zero. Consequently, the proposed algorithm which is direct, efficient and powerful, can solve more general exact solutions with 29 different forms of solutions, including 6 trigonometric function solutions, 4 hyperbolic function solutions, 7 rational function solutions, and 12 exponential function solutions. All exact solutions and symbolic computations in this work have been checked by substituting back into the CRWP equation with the aid of the Maple program.

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