Solving the two-dimensional Hammerstein integral equations using Taylor-series expansion method

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Abstract

Several physical problems can be formulated in terms of either differential or integral equations. A well-known nonlinear integral equation is a Hammerstein equation which is reformulation of a boundary value problem with nonlinear boundary terms. In this research, we develop a Taylor-series expansion method for solving the two-dimensional Hammerstein equation. It was originally using to solve the one-dimensional Hammerstein equation. The objective of study is to find the numerical solutions of the two-dimensional Hammerstein equation using the Taylor-series expansion method. It gives highly accurate approximations for the solution up to the order of the Taylor-series approximation used in the method. Finally, numerical example is presented to illustrate the efficiency and accuracy of this method.

Keywords: Two-dimensional Hammerstein integral equation, Taylor-series expansion method, transform technique.


1 Introduction

Integral equations play an important role as mathematical models for various physical situation such as mechanics, fluid mechanics, elasticity theory and radiative transfer \cite{8}. Integral equations may arise from the ordinary or partial differential equations. The problems with one variable can be formulated as the one-dimensional integral equations whereas some problems with multivariables can also be formulated as the multivariate integral equations. Integral equations can be classified as linear or nonlinear. Fredholm and Volterra integral equations are the most well-known linear integral equation which related to boundary and initial value problems, respectively. Alternatively, the Hammerstein integral equation is a famous nonlinear integral equation. The Hammerstein equation arises naturally as an integral equation reformulation of a

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class of two-point boundary value problem with nonlinear boundary terms. Some more details about the multivariable integral equation can see in [1].

Many researchers have studied and solved the integral equations to find analytical solutions via the analytical methods such as Adomian decomposition method, the method of successive approximations, the series solution method. However, to find the analytical solutions of integral equations might be complicated then the numerical methods have developed to find approximate solutions of integral equations. The well-known numerical methods for solving the Fredholm and Volterra equations are Galerkin method, collocation method, or Nystrom method (see, e.g., [1], [2] and the references cited within). For these methods, a corresponding system of linear equations is produced and the computational cost is high when the matrix is large and dense. In 1999, Yuhe Ren et al. [9] introduced a Taylor-series expansion method to solve the Fredholm and Volterra integral equations numerically. In 2010 and 2011, Huabsomboon et al. ([3], [4]) improved Rens technique to find the numerical solutions of Fredholm and Volterra integral equations, respectively. They have reported that the numerical results give higher accuracy than those reported in [9]. In 2012, Huabsomboon et al. [5] presented a Taylor-series expansion method to solve the one-dimensional Hammerstein equation. These paper also modified the original Taylor-series expansion method using transform technique. A resulting system of nonlinear equations is produced and the computational cost is reduced. In 2012, Novaprateep et al. [6] extended the idea of solving one-dimensional Taylor-series expansion method to solve the multivariate Fredholm equations of the second kind. In our paper, we will find the numerical solutions of the two-dimensional Hammerstein equation using the Taylor-series expansion method by combining the ideas of [5] and [6]. The two-dimensional Hammerstein equation is generally written as

\[
f(t_1, t_2) - \lambda \int_a^b \int_c^d k(t_1, t_2, s_1, s_2) \psi(s_1, s_2, f(s_1, s_2)) \, ds_1 \, ds_2 = g(t_1, t_2), \quad (t_1, t_2) \in [a, b] \times [c, d]
\]

where \(k, g, \psi\) are known functions and \(f\) is unknown function to be determined. Usually, \(k\) is called the kernel, \(g\) is an external forcing term, \(\psi\) is a nonlinear term and \(\lambda\) is numerical parameter which is composed of physical quantities.

This paper is organized as follow. In Section 2, some definitions and basic background are introduced. In Section 3, we present the two-dimensional Hammerstein equation which first transform equation (2.1) into another nonlinear equation. Then a Taylor-series expansion method with two variables and Newton-Raphson method are applied for solving the corresponding nonlinear system. In Section 4, the numerical example is implemented with specific case. Finally, the conclusion is shown.

2 Basic background

**Definition 2.1** (Single variable Taylor series). Let \(f\) be an infinitely differentiable function in some open interval around \(x = a\).

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots
\]

**Definition 2.2** (Multivariable Taylor series). Let \(f\) be an infinitely differentiable function in some open neighborhood around \((x, y) = (a, b)\).

\[
f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
+ \frac{1}{2!} \left[ f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(y - b)^2 \right] + \ldots
\]
\[ \textbf{Definition 2.3 (Jacobian matrix).} \] For any \( \mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n \) is determined by

\[
\mathbf{F}(\mathbf{x}) = \begin{bmatrix}
    f_1(\mathbf{x}) \\
    f_2(\mathbf{x}) \\
    \vdots \\
    f_n(\mathbf{x})
\end{bmatrix}
\]

where \( \mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^T \). Then the Jacobian matrix \( \mathbf{J} \) is defined by

\[
\mathbf{J}(\mathbf{x}) = \begin{bmatrix}
    \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\
    \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n}
\end{bmatrix}.
\]

\[ \text{Newton-Raphson method for multivariate} \]
Consider the system of nonlinear equations \( \mathbf{F}(\mathbf{x}) = 0 \) where \( \mathbf{F} \) and \( \mathbf{x} \) defined by Definition 2.3. The Newton-Raphson iterative is

\[
\mathbf{x}_{N+1} = \mathbf{x}_N - \mathbf{J}^{-1}(\mathbf{x}_N)\mathbf{F}(\mathbf{x}_N), \quad \text{for } N = 0, 1, \ldots
\]

(2.1)

with appropriate initial guess \( \mathbf{x}_0 \). The iterations is terminated with a chosen stopping criteria. Here, the stopping criteria is \( \|\mathbf{x}_{N+1} - \mathbf{x}_N\|_\infty \leq \delta \), where \( \delta \) is tolerance.

3 A Taylor-Series Expansion Method for Two-Dimensional Hammerstein Equation

Consider the two-dimensionl Hammerstein equation written as

\[
f(t_1, t_2) = \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2) \psi(s_1, s_2, f(s_1, s_2)) \, ds_1 \, ds_2 = g(t_1, t_2), \quad (t_1, t_2) \in [0, 1] \times [0, 1].
\]

(3.1)

From (1.1), for simplicity, let \( \lambda = 1 \) and consider the domain only \([0, 1] \times [0, 1]\) since it can be transformed to \([a, b] \times \, [c, d]\).

Using the transform technique similar to [5], by defining

\[
z(s_1, s_2) = \psi(s_1, s_2, f(s_1, s_2)).
\]

(3.2)

Substituting (3.2) into (3.1), we get

\[
f(t_1, t_2) = \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2) z(s_1, s_2) \, ds_1 \, ds_2 = g(t_1, t_2)
\]

or

\[
f(t_1, t_2) = g(t_1, t_2) + \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2) z(s_1, s_2) \, ds_1 \, ds_2.
\]

(3.3)

After substituting (3.3) into (3.2), we obtain

\[
z(t_1, t_2) = \psi(t_1, t_2, g(t_1, t_2) + \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2) z(s_1, s_2) \, ds_1 \, ds_2).
\]

(3.4)

From this transform technique, we see that the double integral term is linear with unknown function \( z \) which appear inside the nonlinear term \( \psi \). It is easier to compute \( z \) from (3.4) than
Newton-Raphson method for multivariate is used with unknown vector \( x \)

\[ z(s_1, s_2) = z(t_1, t_2) + \frac{\partial z(t_1, t_2)}{\partial t_1}(s_1 - t_1) + \frac{\partial z(t_1, t_2)}{\partial t_2}(s_2 - t_2) \]

\[ + \frac{\partial^2 z(t_1, t_2)}{\partial t_1^2} \frac{(s_1 - t_1)^2}{2!} + \frac{\partial^2 z(t_1, t_2)}{\partial t_1 \partial t_2}(s_1 - t_1)(s_2 - t_2) + \frac{\partial^2 z(t_1, t_2)}{\partial t_2^2} \frac{(s_2 - t_2)^2}{2!} + \ldots \]  

Next, the process of Taylor-series expansion method for order \( r \) is presented.

### 3.1 Taylor-Series Expansion Method of Order 1

The system of nonlinear equations to solve for unknown \( z \), \( \frac{\partial z}{\partial t_1} \) and \( \frac{\partial z}{\partial t_2} \) is obtained by following steps. Firstly, substituting (3.5), using only first derivative, into (3.4) and simplifying, we get

\[ z(t_1, t_2) \approx \psi \left( t_1, t_2, g(t_1, t_2) + z(t_1, t_2) \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)ds_1 ds_2 \right) \]

\[ + \frac{\partial z}{\partial t_1} \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)(s_1 - t_1)ds_1 ds_2 + \frac{\partial z}{\partial t_2} \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)(s_2 - t_2)ds_1 ds_2 \].

Differentiating (3.4) with respect to \( t_1 \),

\[ \frac{\partial z}{\partial t_1} = \frac{\partial}{\partial t_1} \psi \left( t_1, t_2, g(t_1, t_2) + \int_0^1 \int_0^1 k((t_1, t_2, s_1, s_2)z(s_1, s_2)ds_1 ds_2 \right) \].

Substituting (3.5), using only first derivative, into (3.7),

\[ \frac{\partial z}{\partial t_1} \approx \frac{\partial}{\partial t_1} \psi \left( t_1, t_2, g(t_1, t_2) + z(t_1, t_2) \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)ds_1 ds_2 \right) \]

\[ + \frac{\partial z}{\partial t_1} \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)(s_1 - t_1)ds_1 ds_2 + \frac{\partial z}{\partial t_2} \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)(s_2 - t_2)ds_1 ds_2 \].

Note that to find \( \frac{\partial}{\partial t_1} \psi \) the chain rule is need.

Similarly, differentiating (3.4) with respect to \( t_2 \) and substituting (3.5), using only first derivative, we obtain

\[ \frac{\partial z}{\partial t_2} \approx \frac{\partial}{\partial t_2} \psi \left( t_1, t_2, g(t_1, t_2) + z(t_1, t_2) \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)ds_1 ds_2 \right) \]

\[ + \frac{\partial z}{\partial t_1} \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)(s_1 - t_1)ds_1 ds_2 + \frac{\partial z}{\partial t_2} \int_0^1 \int_0^1 k(t_1, t_2, s_1, s_2)(s_2 - t_2)ds_1 ds_2 \].

From (3.6), (3.8) and (3.9), the corresponding nonlinear system will be obtained to solve for unknown \( z \), \( \frac{\partial z}{\partial t_1} \) and \( \frac{\partial z}{\partial t_2} \). Then it can be solved by any nonlinear solver scheme. Here, the Newton-Raphson method for multivariate is used with unknown vector \( x = \begin{bmatrix} z \frac{\partial z}{\partial t_1} \frac{\partial z}{\partial t_2} \end{bmatrix} ^T \).

Let \( F(x) = \begin{bmatrix} f_1(x) f_2(x) f_3(x) \end{bmatrix} ^T \) where \( f_1, f_2 \) and \( f_3 \) is defined by moving RHS to LHS from (3.6), (3.8) and (3.9), respectively. Then using the Newton-Raphson iterative (2.1) to solve for unknown \( x_N \) with appropriate initial guess \( x_0 \) and specific tolerance.

Finally, \( z \) is known when \( x_N \) is found. The numerical solutions \( f \) is found by substituting \( z \) into (3.3) and solving it.
Table 1: Computational results with $r = 5$.

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<th>$t_1$</th>
<th>$t_2$</th>
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</table>

3.2 Taylor-Series Expansion Method of Order $r$

For the Taylor-series expansion method of order 1, we see that the nonlinear system of three equations with three unknowns is obtained. It can be solved for $z$, $\frac{\partial z}{\partial t_1}$, and $\frac{\partial z}{\partial t_2}$.

For the Taylor-series expansion method of order 2, the nonlinear system of six equations with six unknowns is obtained. It can be solved for $z$, $\frac{\partial z}{\partial t_1}$, $\frac{\partial z}{\partial t_2}$, $\frac{\partial^2 z}{\partial t_1^2}$, $\frac{\partial^2 z}{\partial t_1 \partial t_2}$, and $\frac{\partial^2 z}{\partial t_2^2}$. Three more equations are needed which come from differentiate (3.4) more one time with respect to $t_1$ or $t_2$. The Taylor series of order 2 is used.

For the Taylor-series expansion method of order $r$, the process can be done in the similar way.

4 Numerical Example

In this section, the numerical example using the Taylor-series expansion method for solving two dimensional Hammerstein equation is presented. The computer programs are run on a personal computer with 2.8 GHz CPU and 8 GB memory.

Example 4.1. Consider the equation

$$f(t_1, t_2) - \int_0^1 \int_0^1 \exp(-t_1 - t_2 - s_1 - s_2)f^2(s_1, s_2)ds_1ds_2 = g(t_1, t_2), \quad t \in [0, 1] \times [0, 1]$$

where $g(t_1, t_2)$ is chosen so that the isolated solution $f(t_1, t_2) = t_1^2 + t_2^2$.

The numerical solutions of the Taylor-series expansion method of order $r$ is computing with initial guess $x_0$ is zero vector and tolerance $\delta = 10^{-12}$. Then we found that the Taylor-series expansion method of order 5 is exact to the machine accuracy, see the results in the Table 1.
In this example, the exponential kernel is used, \( \exp(-t_1 - t_2 - s_1 - s_2) \), and the solution is taken to be a polynomial of degree 2. From Table 1, we see that the numerical solutions of Taylor-series expansion method of order 5 give highly accurate approximation.

5 Conclusion

In this paper, we have extended the idea of the Taylor-series expansion method from one-dimensional Hammerstein equation to two-dimensional Hammerstein equation. Then we have applied it to find the numerical solution of two-dimensional Hammerstein equation. This method can find an approximate solutions with the accuracy depend on the order of Taylor series used. The results of numerical experiment has shown the efficiency and accuracy of this method.

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