The achromatic numbers of unitary addition Cayley graphs

Puttipong Momrit† and Chanon Promsakon‡
Department of Mathematics, Faculty of Science
King Mongkut’s University of Technology North Bangkok, Bangkok 10800, Thailand

Abstract

A complete coloring of a graph $G$ is a proper vertex coloring such that every pair of distinct colors appears on the ends of some edge. The achromatic number of a graph $G$, denoted by $\psi(G)$, is the greatest number of colors in such a coloring.

Let $n$ be a positive integer greater than 1, $\mathbb{Z}_n$ is the integer modulo $n$, $U_n$ is a set of all units in $\mathbb{Z}_n$. The unitary addition Cayley graph, denoted by $G_n$ or $\text{Cay}^+(\mathbb{Z}_n, U_n)$ is a graph whose vertex set is $\mathbb{Z}_n$ and vertices $u$ and $v$ are adjacent if and only if $u + v \in U_n$.

In this paper, we focus on complete coloring properties of unitary addition Cayley graphs. We also find exact values and bounds of the achromatic numbers of some types of $G_n$. Moreover, we give examples to illustrate.

Keywords: Complete coloring, Achromatic number, Units, Addition Cayley graphs.

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1 Introduction

Throughout this paper, we consider only finite, simple, undirected graphs. For standard terminology and notation in graph theory we refer to West [1] and in algebraic graph theory we refer to Biggs [2]. A coloring of a graph is an assignment of colors to the vertices of $G$ such that whenever two vertices are adjacent they are colored differently, a $k$-coloring of $G$ uses $k$ colors. In a complete $k$-coloring, for every pair of distinct colors, there exist two adjacent vertices to which these two colors are assigned. If a graph $G$ has a complete $k$-coloring, then $G$ must contain at least $\binom{k}{2}$ edges. Consequently, if the size of a graph $G$ is less than $(\frac{k}{2})$ for some positive integer $k$, then $G$ can not have a complete $k$-coloring.

The chromatic number of a graph $G$, denoted by $\chi(G)$, is the smallest positive integer $k$ such that $G$ has a $k$-coloring. The achromatic number of $G$, denoted by $\psi(G)$, is the largest positive integer $k$ for which there is a complete $k$-coloring of $G$. In contrast to the chromatic number, which is arguably the most studied graph parameter, the achromatic number was first introduced by Harary and Hedetniemi in 1970 [3]. It therefore follows that

$$\psi(G) \geq \chi(G)$$

for every graph $G$. Certainly, if $G$ is a graph of order $n$, then $\psi(G) \leq n$. and for the complete graph $K_n$, $\psi(K_n) = \chi(K_n) = n$. Hence there are graphs $G$ for which $\psi(G) = \chi(G)$.

†Corresponding author.
‡Speaker.
E-mail address: chai_ny@hotmail.com (P. Momrit), chanon.p@sci.kmutnb.ac.th (C. Promsakon).
For example, we have that the 3-coloring of the path $P_4$ is a complete 3-coloring and the 4-coloring of the path $P_8$ is a complete 4-coloring. Furthermore, while $\chi(P_4) = \chi(P_8) = 2$ for the two paths $P_4$ and $P_8$ shown in Figure 1, $\psi(P_4) = 3$ and $\psi(P_8) = 4$. Let’s see why $\psi(P_8) = 4$. First, $\psi(P_8) \geq 4$ since the 4-coloring shown in Figure 1 is a complete 4-coloring, while $\psi(P_8) < 5$ since the size of $P_8$ is 7 which is less than $(\frac{7}{2}) = 10$. Thus $\psi(P_8) = 4$.

![Figure 1: A complete 3-coloring of $P_4$ and a complete 4-coloring of $P_8$](image)

The complement $\bar{G}$ of a graph $G$ is the graph whose vertex set is $V(G)$ and where $uv$ is an edge of $\bar{G}$ if and only if $uv$ is not an edge of $G$. Observe that if $G$ is a graph of order $n$ and size $m$, then $\bar{G}$ is a graph of order $n$ and $\binom{n}{2} - m$. For example, a graph $G_1$ in Figure 2 is a complement graph of $K_2$.

The join of two graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, of two vertex-disjoint graphs $G_1$ and $G_2$ has $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$ 

Let graphs $G_1$ and $G_2$ be the graphs shown in Figure 2. We have the vertex set and the edge set

$$V(G_1) = \{u_1, u_2\}, E(G_1) = \emptyset$$

and

$$V(G_2) = \{v_1, v_2, v_3\}, E(G_2) = \{v_1v_2, v_2v_3\}.$$

So the join graph $G_1 + G_2$ has the vertex set $V(G_1 + G_2) = \{u_1, u_2, v_1, v_2, v_3\}$ and the edge set $E(G_1 + G_2) = \{u_1v_1, u_1v_2, u_1v_3, u_2v_1, u_2v_2, u_2v_3, v_1v_2, v_1v_3\}$.

![Figure 2: A join graph](image)

A graph $G$ is a bipartite graph if it is possible to partition $V(G)$ into two subsets $U$ and $W$, such that every edge of $G$ joins a vertex of $U$ and a vertex of $W$. A bipartite graph is a complete bipartite graph if every vertex of $U$ is adjacent to every vertex of $W$. If the sets $U$ and $W$ of a complete bipartite graph contain $s$ and $t$ vertices, then this graph is denoted by $K_{s,t}$ or $K_{t,s}$. Some examples are shown in Figure 5. More generally, for an integer $k \geq 2$ and positive integers $n_1, n_2, ..., n_k$, a complete $k$-partite graph denoted by $K_{n_1,n_2,...,n_k}$ is the graph $G$ whose vertex set can be partitioned into $k$ subsets $V_1, V_2, ..., V_k$ with $|V_i| = n_i$ for $1 \leq i \leq k$ such that $uv \in E(G)$ if $u \in V_i$ and $v \in V_j$, where $1 \leq i, j \leq k$ and $i \neq j$. The two graphs in Figure 6 are complete 3-partite graphs.

For a positive integer $n > 1$, let $U_n$ be the set of all units of $\mathbb{Z}_n$. The unitary Cayley graph of $\mathbb{Z}_n$, denoted by $X_n$ is the graph whose vertex set is $\mathbb{Z}_n$, the integers modulo $n$ and two vertices $u, v$ are adjacent if and only if $u - v \in U_n$. The unitary Cayley graph $X_n$ is also defined as $X_n = Cay(\mathbb{Z}_n, U_n)$. Examples of $X_n$ are shown in Figure 3.
In 2007, Klotz and Sander [4] studied properties of unitary Cayley graphs. They determined some properties of unitary Cayley graphs such as the chromatic number, the clique number, the independence number, the diameter and the vertex connectivity. And a year later, Boggess et al. explored structural properties of unitary Cayley graphs, including the clique number, the chromatic number, the vertex connectivity, the edge connectivity, the planarity, and the crossing number in [5].

Next, we introduce the unitary addition Cayley graphs. For a positive integer \( n > 1 \), let \( U_n \) be the set of all units of \( \mathbb{Z}_n \). The unitary addition Cayley graph of \( \mathbb{Z}_n \), denoted by \( G_n \), is the graph whose vertex set is \( \mathbb{Z}_n \) and two vertices \( u, v \) are adjacent if and only if \( u + v \in U_n \). The unitary addition Cayley graph \( G_n \) is also defined as, \( G_n = Cay^+(\mathbb{Z}_n, U_n) \). Some examples of unitary addition Cayley graphs are displayed in Figure 4.

Sinha et al. [6] studied properties of unitary addition Cayley graphs in 2011. They discussed several properties of unitary addition Cayley graphs and also obtain the characterization of planarity and outerplanarity of unitary addition Cayley graphs.

In 2015, Palanivel and Chithra [7] studied structure of unitary addition Cayley graphs. They determined some structure properties of unitary addition Cayley graphs such as the independence number, the chromatic number, the edge chromatic number, the diameter, the vertex connectivity, the edge connectivity and the perfectness.

In this work, we are interested in studying complete \( k \)-coloring properties of unitary addition Cayley graphs. We show exact values and bounds of the achromatic numbers of some cases of the unitary addition Cayley graph \( G_n \).

2 Preliminaries

We now give some results which are essentials and important in this study.

Theorem 2.1. [3] Every complete bipartite graph has achromatic number 2.

According to Theorem 2.1, we have \( \psi(K_{2,3}) = \psi(K_{3,3}) = 2 \). A complete 2-coloring of both graphs are given in Figure 5.
The following information of the achromatic number of a join graph was obtained by Harary and Hedetniemi.

**Proposition 2.2.** [3] If $G = G_1 + G_2$, then

$$\psi(G) = \psi(G_1) + \psi(G_2).$$

An example is shown in Figure 2. Because $G_1$ has 1-complete coloring, $\psi(G_1) = 1$. Since $G_2$ is a complete bipartite graph, we have $\psi(G_2) = 2$. By Proposition 2.2,

$$\psi(G) = \psi(G_1) + \psi(G_2) = 1 + 2 = 3.$$

While Proposition 2.2 shows the achromatic numbers of a join graph, Xu [8] established an upper bound for $\psi(G) - \chi(G)$ in terms of the order of $G$.

**Theorem 2.3.** [8] For every graph $G$ of order $n \geq 2$,

$$\psi(G) - \chi(G) \leq \frac{n}{2} - 1.$$

The achromatic numbers of all cycles was determined by Hell and Miller [9].

**Theorem 2.4.** [9] For each $n \geq 3$, $\psi(C_n) = \max\{k|k\lfloor \frac{n}{2} \rfloor \leq n\} - s(n)$, where $s(n)$ is the number of positive integer solutions of $n = 2x^2 + x + 1$. (Note that $s(n) = 0$ or 1).

Examples for the above theorem are shown in the next section.

Some properties of the unitary addition Cayley graph used in this work are as follow. Let $\phi(n)$ denote the well known Euler’s totient function, which gives the number of numbers less than $n$ that are co-prime to $n$. The proof of the following can be seen in [6]. Degree of vertices in unitary addition Cayley graphs $G_n$ are shown in the next theorem.

**Theorem 2.5.** [6] Let $v$ be any vertex of the unitary addition Cayley graph $G_n$. Then

$$d(v) = \begin{cases} 
\phi(n) & \text{if } n \text{ is even}, \\
\phi(n) & \text{if } n \text{ is odd and } (v,n) \neq 1, \\
\phi(n) - 1 & \text{if } n \text{ is odd and } (v,n) = 1.
\end{cases}$$

From the previous theorem, the total number of edges in the unitary addition Cayley graphs $G_n$ can be computed.

**Corollary 2.6.** [6] The total number of edges in the unitary addition Cayley graphs $G_n$ is

$$|E(G_n)| = \begin{cases} 
\frac{1}{2}n\phi(n) & \text{if } n \text{ is even}, \\
\frac{1}{2}(n-1)\phi(n) & \text{if } n \text{ is odd}.
\end{cases}$$

The condition for the unitary addition Cayley graphs $G_n$ being bipartite are shown in the next theorem.
Theorem 2.7. [6] The unitary addition Cayley graph $G_n, n \geq 2$ is bipartite if and only if either $n$ is even or $n = 3$.

For example, the unitary addition Cayley graph $G_5$ (see Figure 4) has vertex set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and $U_5 = \{1, 2, 3, 4\}$. It implies that $\phi(5) = |U_5| = 4$. According to Theorem 2.5, we get that $d(1) = d(2) = d(3) = d(4) = 3$ and $d(0) = 4$. From Corollary 2.6, the total number of edges in $G_5$ is $\frac{1}{2}(5 - 1)(4) = 8$. Moreover, in Figure 4, we see that $G_3$ and $G_4$ are bipartite graphs which agrees with Theorem 2.7.

We end this section by giving an exact value of achromatic number of complete $r$-partite graphs as follow.

Lemma 2.8. For each $r \geq 2$, every complete $r$-partite graph has the achromatic number $r$.

Proof. We prove by induction. Let $K_{n_1,n_2,\ldots,n_r}$ be a complete $r$-partite graph for positive integer $r \geq 2$. Let $r = 2$. We get $K_{n_1,n_2}$ and from Theorem 2.1, $\psi(K_{n_1,n_2}) = 2$. Next, we show the induction step. Suppose that $\psi(K_{n_1,n_2,\ldots,n_k}) = k$. Since $K_{n_1,n_2,\ldots,n_k,n_{k+1}} = K_{n_1,n_2,\ldots,n_k} + \tilde{K}_{n_{k+1}}$, by Proposition 2.2,

$$\psi(K_{n_1,n_2,\ldots,n_k,n_{k+1}}) = \psi(K_{n_1,n_2,\ldots,n_k}) + \psi(\tilde{K}_{n_{k+1}}) = k + 1.$$  

This completes the proof. □

For example, we consider the achromatic number of graphs in Figure 6. By Lemma 2.8, we get that $\psi(K_{2,2,2}) = \psi(K_{1,2,3}) = 3$.

![Figure 6: Some examples for complete 3-partite graphs.](image)

3 Main Results

We now give some results of the achromatic numbers in the unitary addition Cayley graphs for some types of integer $n$.

Theorem 3.1. For every positive integer $k$, $\psi(G_{2k}) = 2$.

Proof. Let $k$ be a positive integer. From Theorem 2.7, $G_{2k}$ is a bipartite graph. We separate $\mathbb{Z}_{2k}$ into $U_{2k}$ and $U'_{2k} = \mathbb{Z}_{2k} \setminus U_{2k}$. Then for any $n \in \mathbb{Z}_{2k}$, if $n$ is an even number, then $n \in U'_{2k}$. Otherwise, $n \in U_{2k}$. We will show that any two vertices in $U_{2k}$ are not adjacent. Let $u, v \in U_{2k}$. Then $u, v$ are odd numbers and $u + v$ is an even number. So $(u + v, 2^k) \neq 1$ and $u + v \notin U_{2k}$. Therefore the vertices $u$ and $v$ are not adjacent. Similarly, any two vertices in $U'_{2k}$ are not adjacent. Because the vertices $u, v \in U'_{2k}$ are even number and $u + v$ is an even number, so $u + v \notin U_{2k}$.

Next, we show that $G_{2k}$ is a complete bipartite graph. Let the vertices $a \in U_{2k}$ and $b \in U'_{2k}$. Since $a$ is an odd number and $b$ is an even number, we have $a + b$ is an odd number. So $(u + v, 2^k) = 1$ and $u + v \in U_{2k}$. Thus any vertices $a \in U_{2k}$ and $b \in U_{2k}$ are adjacent. So $G_{2k}$ is a complete bipartite graph and hence $\psi(G_{2k}) = 2$, by Theorem 2.1. □

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For example, we consider the chromatic number of $G_4$ and $G_8$ shown in Figure 7. We see that $G_4$ and $G_8$ are complete bipartite graphs. By Theorem 3.1, we get that $\psi(G_4) = \psi(G_8) = 2$.

**Theorem 3.2.** For each $k \geq 1$, let $p$ be an odd prime number. Then

$$\psi(G_{pk}) = 1 + \frac{\phi(p^k)}{2}.$$  

**Proof.** Let $k$ be a positive integer. We separate $\mathbb{Z}_{pk}$ into $U_{pk}$ and $U'_{pk} = \mathbb{Z}_{pk} \setminus U_{pk}$. Then for any $x \in \mathbb{Z}_{pk}$, if $p \mid x$, then $x \in U'_{pk}$. Otherwise, $x \in U_{pk}$. We set $A_i = \{x \in \mathbb{Z}_{pk} | x \equiv i \pmod{p}\}$ for $i = 0, 1, 2, ..., p - 1$.

We claim that $G_{pk} = H_0 + H_1 + ... + H_{p-1}$ where $H_j$ is the induced subgraph of $G_{pk}$ whose vertex set is $A_j \cup A_{p-j}$ for $j = 0, 1, 2, ..., \frac{p-1}{2}$. Let $u$ and $v$ be adjacent vertices in $G_{pk}$. Then $u + v \in U_{pk}$. By division Algorithm, we have $u = q_1 p + m$ and $v = q_2 p + n$ where $m, n, q_1, q_2 \in \mathbb{Z}$ and $0 \leq m \leq p - 1$ and $0 \leq n \leq p - 1$. So $u \in A_m$ and $v \in A_n$. Since $u + v \in U_{pk}$, we have $p \nmid (u + v)$. This implies that $u \neq p - v \pmod{p}$ and $u \in V(H_m)$ and $v \in V(H_n)$. Hence $u$ and $v$ are adjacent in $H_0 + H_1 + ... + H_{p-1}$.

Next, let the vertices $x \in H_i$ and $y \in H_j$ where $i \neq j$. There are four cases to be considered.

**Case I:** $x \equiv i \pmod{p}$ and $y \equiv j \pmod{p}$. Then $x + y \equiv i + j \pmod{p}$. Since $0 \leq i + j \leq 2p - 2$ and $i \neq p - j \pmod{p}$, we conclude that $x + y \equiv i + j \neq 0 \pmod{p}$. Hence $x$ and $y$ are adjacent vertices in $G_{pk}$.

**Case II:** $x \equiv i \pmod{p}$ and $y \equiv p - j \pmod{p}$. Then $x + y \equiv i + p - j \neq 0 \pmod{p}$. Hence $x$ and $y$ are adjacent vertices in $G_{pk}$.

**Case III:** $x \equiv p - i \pmod{p}$ and $y \equiv j \pmod{p}$. Similar to case II, we conclude that $x$ and $y$ are adjacent vertices in $G_{pk}$.

**Case IV:** $x \equiv p - i \pmod{p}$ and $y \equiv p - j \pmod{p}$. So $x + y \equiv p - i + p - j = -i + j \neq 0 \pmod{p}$. Hence $x$ and $y$ are adjacent vertices in $G_{pk}$.

Therefore $E(H_0 + H_1 + ... + H_{p-1}) \subseteq E(G_{pk})$ and hence they are the same graph.

We will find the chromatic number of $G_{pk}$. Let $u_1, v_1 \in H_0$. Then $p \mid u_1$ and $p \mid v_1$. That implies $p \mid (u_1 + v_1)$ and $(u_1 + v_1, pk) \neq 1$. So $u_1 + v_1 \notin U_{pk}$ and $u_1, v_1$ are not adjacent for all $u_1, v_1 \in H_0$. That is $\psi(H_0) = 1$.

Next, we will show that $\psi(H_s) = p^{k-1}$ for $s = 1, 2, ..., \frac{p-1}{2}$. We first show that the subgraph induced by $A_s$ is a complete graph. Let vertices $u_2, v_2 \in A_s$, that is $u_2 + v_2 \equiv s + s \equiv 2s \pmod{p}$. Since $p \mid 2s$, that implies $p \nmid (u_2 + v_2)$. So $u_2 + v_2 \in U_{pk}$ for all $u_2, v_2 \in A_s$. Therefore each $A_s$ for $s = 1, 2, ..., \frac{p-1}{2}$ induces a complete graph of order $p^{k-1}$. Then we show that $H_s$ is a disconnected subgraph composed of two components: one induced by $A_s$ and the other induced by $A_{p-s}$ for any $s = 1, 2, ..., \frac{p-1}{2}$. Let the vertices $u_3 \in A_s$ and the vertices $v_3 \in A_{p-s}$. Then we get that $u_3 + v_3 \equiv s + p - s \equiv 0 \pmod{p}$. Since $p \nmid (u_3 + v_3)$, that implies $u_3 + v_3 \notin U_{pk}$. So $u_3$ and $v_3$ are not adjacent. Therefore $H_s$ is disconnected for any $s = 1, 2, ..., \frac{p-1}{2}$. By the fact that
the achromatic number of a complete graph with order \( n \) is \( n \), we conclude that \( \psi(H_s) = p^{k-1} \) for \( s = 1, 2, \ldots, \frac{p-1}{2} \).

Form Proposition 2.2, we get that
\[
\psi(G_{p^k}) = \psi(H_0) + \psi(H_1) + \psi(H_2) + \ldots + \psi(H_{\frac{p-1}{2}})
= 1 + p^{k-1} + p^{k-1} + \ldots + p^{k-1}
= 1 + p^{k-1} \left( \frac{p-1}{2} \right)
= 1 + \frac{p^{k-1} - 1}{2}.
\]
Hence, \( \psi(G_{p^k}) = 1 + \frac{\phi(p^k)}{2} \).

For example, we consider the achromatic number of \( G_9 \) in Figure 8. By following the proof of Theorem 3.2, we set \( A_0 = \{0, 3, 6\} \), \( A_1 = \{1, 4, 7\} \) and \( A_2 = \{2, 5, 8\} \). We let \( G_9 = H_0 + H_1 \) where \( H_0 \) has vertex set \( A_0 \) and \( H_1 \) has vertex set \( A_1 \cup A_2 \), as follow in Figure 8. Next we give colors to \( H_0 \) has one color and \( H_1 \) has three colors. Since \( G_9 = H_0 + H_1 \), then \( \psi(G_9) = \psi(H_0) + \psi(H_1) \). Therefore \( \psi(G_9) = 4. \)

![Diagram of G9 and its induced subgraphs H0, H1.](image)

**Figure 8:** The unitary addition Cayley graph \( G_9 \) and its induced subgraphs \( H_0, H_1 \).

**Theorem 3.3.** Let \( p \) be an odd prime number. Then
\[ p \leq \psi(G_{2p}) \leq p + 1. \]

**Proof.** We separate \( \mathbb{Z}_{2p} \) into \( U_{2p} \) and \( U'_{2p} = \mathbb{Z}_{2p} \setminus U_{2p} \). Then for any odd number \( n \) of \( \mathbb{Z}_{2p} \), if \( p \mid n \), then \( n \in U'_{2p} \). Otherwise, \( n \in U_{2p} \). We separate \( U'_{2p} \) into \( A = \{0, p\} \) and \( A' = U'_{2p} \setminus A \). We set \( B_i = C_i \cup C_i' \) such that \( C_i = \{u_i \in U_{2p} | u_i \equiv i \pmod{p}\} \) and \( C_i' = \{v_i \in A' | v_i \equiv p - u_i \pmod{p}\} \) for all \( i = 1, 2, \ldots, p - 1 \). It is clearly that \( |B_i| = 2 \) and implies that set \( C_i \) contains an odd number that is not \( p \). Similarly, set \( C_i' \) contains an even number that is not 0.

We define a coloring \( f : \mathbb{Z}_{2p} \to \{0, 1, 2, \ldots, p - 1\} \) by
\[
f(x) = \begin{cases} 
0 & \text{if } x \in A, \\
i & \text{if } x \in B_i,
\end{cases}
\]
for all \( x \in \mathbb{Z}_{2p} \) where \( i = 1, 2, \ldots, p - 1 \). We will show that \( f \) is a proper coloring by assuming vertices \( u \) and \( v \) are elements in \( \mathbb{Z}_{2p} \) such that \( f(u) = f(v) \). Then there are two cases to be considered.

**Case I:** Both \( u \) and \( v \) are in \( A \). Then \( p \mid u \) and \( p \mid v \). That implies \( p \mid (u + v) \) and \( (u + v, 2p) \neq 1 \). So \( u + v \notin U_{2p} \). Hence \( u \) and \( v \) are not adjacent in \( G_{2p} \).

**Case II:** Both \( u \) and \( v \) are in \( B_i \) and let \( u \in C_i \) and \( v \in C_i' \). Then \( u \equiv i \pmod{p} \) and \( v \equiv p - u \equiv p - i \pmod{p} \). So \( u + v \equiv i + p - i \equiv p \pmod{p} \). This implies that \( p \mid (u + v) \) and \( (u + v, 2p) \neq 1 \). So \( u + v \notin U_{2p} \). Hence \( u \) and \( v \) are not adjacent in \( G_{2p} \). By two cases, we conclude that \( f \) is proper coloring. Therefore \( \psi(G_{2p}) \geq p \).
By Theorem 2.7, we have the chromatic number \( \chi(G_{2p}) = 2 \). Since the order of \( G_{2p} \) is \( 2p \), by Theorem 2.3,
\[
\psi(G_{2p}) - \chi(G_{2p}) \leq \frac{2p}{2} - 1 \\
\psi(G_{2p}) - 2 \leq p - 1 \\
\psi(G_{2p}) \leq p + 1.
\]
Therefore \( p \leq \psi(G_{2p}) \leq p + 1 \). \( \square \)

For example, let consider the achromatic number of \( G_{10} \) in Figure 9. By following the proof of Theorem 3.3, we have \( U'_{10} = \{0, 2, 4, 5, 6, 8\} \). We set \( A = \{0, p\} \) and
\[
B_1 = C_1 \cup C'_1, \quad B_2 = C_2 \cup C'_2, \quad B_3 = C_3 \cup C'_3, \quad B_4 = C_4 \cup C'_4.
\]
where \( C_1 = \{1\}, C_2 = \{7\}, C_3 = \{3\}, C_4 = \{9\}, C'_1 = \{4\}, C'_2 = \{8\}, C'_3 = \{2\}, \) and \( C'_4 = \{6\} \). So we get that \( A = \{0, 5\}, B_1 = \{1, 4\}, B_2 = \{7, 8\}, B_3 = \{3, 2\}, \) and \( B_4 = \{9, 6\} \). Then we refigure \( G_{10} \) in the form of 5-partite as shown in Figure 9. We give distinct colors to each partite. So \( \psi(G_{10}) \geq 5 \). And by Theorem 3.3, \( \psi(G_{10}) \leq 6 \).

![Figure 9: The unitary addition Cayley graph \( G_{10} \) and \( G_{10} \) as 5-partite.](image)

According to Theorem 3.3, the achromatic number of \( G_6 \) in Figure 10 is \( 3 \leq \psi(G_6) \leq 4 \). We refigure \( G_6 \) in the form of cycle of order 6 as shown in Figure 10 and then \( \psi(G_6) = 3 \), by Theorem 2.4. Let’s see why \( \psi(G_6) = 3 \). First, find \( s(6) \), the number of positive integer solutions of equation \( 6 = 2x^2 + x + 1 \). That is \( s(n) = 0 \). Then \( \psi(G_6) = \max\{k|k\lfloor \frac{k}{2}\rfloor \leq n\} - s(n) \). When \( k = 3 \), \( \psi(G_6) = \max\{3\lfloor \frac{3}{2}\rfloor \leq 6\} - 0 = 3 \) and when \( k = 4, 4\lfloor \frac{4}{2}\rfloor \leq 6 \). Therefore \( \psi(G_6) = 3 \).

![Figure 10: The unitary addition Cayley graph \( G_6 \) and \( G_6 \) as cycle of order 6.](image)
4 Conclusion

In this paper, we get the exact values of achromatic number of unitary addition Cayley graphs of order in the form power of prime number. Additionally, we also get the bounds of achromatic number of unitary addition Cayley graphs of order in the form product of 2 with odd prime number.

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