Regularity of some transformation semigroups with restricted range

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Abstract

For a semigroup \( S \), we call an element \( a \) in \( S \) regular if there exists an element \( x \) in \( S \) such that \( a = axa \). If every element in \( S \) is regular, then \( S \) is called a regular semigroup. Given a nonempty set \( X \), we denote by \( AM(X) \) the almost one-to-one transformation semigroup on \( X \), by \( AE(X) \) the almost onto transformation semigroup on \( X \), by \( OM(X) \) the opposite semigroup of one-to-one transformation semigroup on \( X \) and by \( OE(X) \) the opposite semigroup of onto transformation semigroup on \( X \). In this paper, we determine the regularity of a generalisation of these semigroups, considered as subsemigroups of \( T(X,Y) \), the semigroup of transformations from \( X \) to a nonempty subset \( Y \) of \( X \). Moreover, regularity of specific elements in \( T(X,Y) \) are characterised.

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1 Introduction

The natural partial order \( \leq \) on a semigroup \( S \) is defined by \( a \leq b \) if and only if \( a = xb = by \) and \( a = ay \) for some \( x, y \in S^1 \) where \( S^1 \) is the semigroup \( S \) if \( S \) contains an identity; otherwise \( S^1 \) is the semigroup obtained from \( S \) by adjoining a new symbol 1 as its identity. It is known that any semigroup endowed with the natural partial order will always pass down the order to its regular subsemigroups; regularity therefore becomes a mainstream subject in the study of the natural partial order on semigroups. Our purpose is to investigate the regularity of certain transformation semigroups with restricted range, some of which can be considered as a generalisation of the semigroups examined in [1].

For a nonempty set \( X \) we let \( T(X) \) be the set of transformations on \( X \). It is clear that, under composition, \( T(X) \) is a semigroup, known as the full transformation semigroup on \( X \). By a regular element in a semigroup \( S \) we mean an element \( a \) that can be expressed as \( axa \) for

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some element $x$ in $S$, and if all elements in $S$ are regular we will call $S$ a regular semigroup. A well-known instance of regular semigroups is the full transformation semigroup. We now introduce a familiar generalisation of the full transformation semigroup. Given a nonempty subset $Y$ of $X$, we let
\[
T(X, Y) = \{ \alpha \in T(X) : \text{ran } \alpha \subseteq Y \}.
\]
This is a subsemigroup of $T(X)$, and one can see that when $Y = X$ we have $T(X, Y) = T(X)$. Also, if $Y$ is a singleton, $T(X, Y)$ is a singleton. Then in these two cases $T(X, Y)$ is regular; however, it fails in the other cases, as stated below.

**Theorem 1.1.** [3] $T(X, Y)$ is regular if and only if $|Y| = 1$ or $Y = X$.

In the rest of this section we introduce other subsemigroups of $T(X)$ related to $T(X, Y)$. Note that throughout this paper all transformations will act on the right-hand side of the argument, and for any element $\alpha$ in $T(X)$, we consider $x\alpha^{-1}$ to be the inverse image of $x$ under $\alpha$.

For any transformation $\alpha$ in $T(X)$, it is almost one-to-one if $\{x \in X : |x\alpha\alpha^{-1}| > 1\}$ is finite. We denote by $AM(X)$ the set of almost one-to-one transformations on $X$. Likewise, there is a generalised notion of surjectivity; a transformation $\alpha$ in $T(X)$ is almost onto if $X \setminus \text{ran } \alpha$ is finite. The set of almost onto transformations on $X$ is denoted by $AE(X)$. Both $AM(X)$ and $AE(X)$ are subsemigroups of $T(X)$, known as the almost one-to-one transformation semigroup on $X$ and the almost onto transformation semigroup on $X$, respectively. Note that $AM(X)$ is regular if $X$ is finite, and vice versa. This is also the case for $AE(X)$.

**Theorem 1.2.** [2, p. 133] Let $X$ be a nonempty set. The following statements are equivalent:

(i) $X$ is finite,
(ii) $AM(X)$ is regular,
(iii) $AE(X)$ is regular.

Now we present a generalisation of $AM(X)$ and $AE(X)$. Let $Y$ be a nonempty subset of $X$. By $AM(X, Y)$ we mean the set of all elements in $AM(X)$ whose range is contained in $Y$. In other words, we have
\[
AM(X, Y) = \{ \alpha \in T(X, Y) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is finite } \}.
\]
Clearly, if $Y = X$, then $AM(X, Y) = AM(X)$. In the same fashion, we have a generalisation of $AE(X)$, defined by
\[
AE(X, Y) = \{ \alpha \in T(X, Y) : X \setminus \text{ran } \alpha \text{ is finite } \}.
\]
The reader should be aware that there is a chance that $AM(X, Y)$ or $AE(X, Y)$ becomes the empty set. When this is not the case, $AM(X, Y)$ and $AE(X, Y)$ are called the semigroup of almost one-to-one transformations on $X$ with restricted range $Y$, and the semigroup of almost onto transformations on $X$ with restricted range $Y$, respectively. We discuss about these semigroups later in the next section.

Next, with an obligation that $X$ is infinite, we have a few more semigroups relevant to those we defined above, namely
\[
OM(X) = \{ \alpha \in T(X) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is infinite } \} \quad \text{and}
\]
\[
OM(X, Y) = \{ \alpha \in T(X, Y) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is infinite } \}.
\]
Clearly, both are subsemigroups of $T(X)$. We call $OM(X)$ the opposite semigroup of one-to-one transformation semigroup on $X$ and $OM(X, Y)$ the opposite semigroup of one-to-one transformation semigroup on $X$ with restricted range $Y$.

Furthermore, under the same constraint that $X$ is infinite, we define
\[
OE(X) = \{ \alpha \in T(X) : X \setminus \text{ran } \alpha \text{ is infinite } \} \quad \text{and}
\]
\[
OE(X, Y) = \{ \alpha \in T(X, Y) : X \setminus \text{ran } \alpha \text{ is infinite } \}.
\]
The former is the opposite semigroup of onto transformation semigroup on \( X \) and the latter is the opposite semigroup of onto transformation semigroup on \( X \) with restricted range \( Y \). It is clear that \( OM(X,Y) \) and \( OE(X,Y) \) respectively are a generalisation of \( OM(X) \) and \( OE(X) \).

Note that if \( Y \) is a singleton, \( OM(X,Y) \) and \( OE(X,Y) \) only contain a constant map; therefore, they are regular. Additionally, there is a possibility that \( AM(X) \) and \( AE(X) \) are regular. But we do not need to worry about \( OM(X) \) and \( OE(X) \), as they definitely are nonregular semigroups.

**Theorem 1.3.** \([2]\) \( OM(X) \) and \( OE(X) \) are not regular.

However, the semigroup \( OM(X) \cap OE(X) \), surprisingly, is always regular. This is also the case for \( AM(X) \cap AE(X) \).

## 2 Main Results

This section is roughly divided into two parts. The first is devoted to the study of regularity of \( AM(X,Y) \) and \( AE(X,Y) \), and the second is concerned with regularity of \( OM(X,Y) \) and \( OE(X,Y) \). First of all, provided below is a necessary and sufficient condition for being a regular element in \( T(X,Y) \).

**Theorem 2.1.** For any transformation \( \alpha \) in \( T(X,Y) \) we have \( \alpha \) is regular if and only if \( Y\alpha = \text{ran} \alpha \). In particular, every injection in \( T(X,Y) \) is not regular when \( Y \) is a proper subset of \( X \).

**Proof.** Here we only prove the first statement as the second follows immediately from the first. To do so, we first assume that \( \alpha \) is a regular element in \( T(X,Y) \). If \( y \) belongs to \( \text{ran} \alpha \), then \( y = x\alpha \) for some \( x \in X \), and by regularity of \( \alpha \), there is a transformation \( \beta \) in \( T(X,Y) \) such that \( \alpha = \alpha \beta \alpha \). Hence, \( y = x\alpha = x\alpha \beta \alpha = (x\alpha \beta \alpha) \in Y\alpha \).

For the other implication, for each \( y \) in \( \text{ran} \alpha \), we choose an element \( z_y \) in \( y\alpha^{-1} \cap Y \) (such an element exists as \( Y\alpha = \text{ran} \alpha \)). Then let \( b \) be an element in \( Y \) and define \( \beta : X \to Y \) by letting \( y\beta = z_y \) if \( y \) is in \( \text{ran} \alpha \); otherwise \( y\beta = b \). Now, for any \( x \in X \), we have \( x\alpha \beta \alpha = (x\alpha)\beta \alpha = z_x \alpha \alpha = x\alpha \), so \( \alpha \beta \alpha = \alpha \). \( \Box \)

As we mentioned before, there is no guarantee that \( AM(X,Y) \) and \( AE(X,Y) \) will be semigroups. We then need a characterisation for \( AM(X,Y) \) and \( AE(X,Y) \) to be semigroups.

**Proposition 2.2.** Let \( X \) be an infinite set. We have

(i) \( AM(X,Y) \) is not the empty set if and only if \( |X| = |Y| \),
(ii) \( AE(X,Y) \) is not the empty set if and only if \( X \setminus Y \) is finite.

**Proof.** To show that (i) holds, we first assume that there exists an element in \( AM(X,Y) \), say \( \alpha \). Then, for any element \( y \) in \( \text{ran} \alpha \) we have that \( y\alpha^{-1} \) is finite. Obviously, the set \( \{y\alpha^{-1} : y \in \text{ran} \alpha \} \) is a partition of \( X \), and therefore, \( \text{ran} \alpha \) must be an infinite set with the same cardinality as \( X \). Then \( |X| = |Y| \). The converse implication follows from the fact that if \( X \) and \( Y \) have the same cardinality then all bijections from \( X \) to \( Y \) belong to \( AM(X,Y) \).

Next, in order to prove the necessity of (ii) we assume that \( AE(X,Y) \) contains a transformation \( \beta \). Then \( X \setminus \text{ran} \beta \) is finite and hence so is \( X \setminus Y \). For the sufficiency, since \( X \) is infinite and \( X \setminus Y \) is finite, \( X \) and \( Y \) clearly have the same cardinality. Hence \( AE(X,Y) \) is not empty, as it at least contains all bijections from \( X \) to \( Y \). \( \Box \)

Below is a consequence of Proposition 2.2.

**Proposition 2.3.** Let \( X \) be an infinite set.

(i) \( AM(X,Y) \) is a semigroup if and only if \( |X| = |Y| \),
(ii) \( AE(X,Y) \) is a semigroup if and only if \( X \setminus Y \) is finite,
(iii) \( AM(X,Y) \cap AE(X,Y) \) is a semigroup if and only if \( X \setminus Y \) is finite.
Proof. It remains to show that $AM(X,Y)$ and $AE(X,Y)$ are closed under composition. The proofs are straightforward and therefore omitted.  

Note that $AM(X,Y) = AM(X)$ and $AE(X,Y) = AE(X)$ when $Y = X$. We next show that semigroups of our interest are different under some constraints.

**Proposition 2.4.** For an infinite set $X$ and a subset $Y$ of $X$ with $X \setminus Y$ finite, neither $AM(X,Y) \setminus AE(X,Y)$ nor $AE(X,Y) \setminus AM(X,Y)$ is the empty set.

Proof. Since $Y$ is infinite, there exists an infinite subset $Z$ of $Y$ with $|Y| = |Z| = |Y \setminus Z|$. Choose an element $z$ in $Z$. Provided two bijections $\varphi : Y \to Y \setminus Z$ and $\psi : Z \to Y \setminus \{z\}$, one can define $\alpha \in AM(X,Y) \setminus AE(X,Y)$ and $\beta \in AE(X,Y) \setminus AM(X,Y)$ by letting

$$x\alpha = \begin{cases} x\varphi & \text{if } x \in Y; \\ z & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} x\psi & \text{if } x \in Z; \\ z & \text{otherwise.} \end{cases}$$

Since $X \setminus \text{ran } \alpha \supseteq Y \setminus \text{ran } \alpha = Z \setminus \{z\}$, we have $\alpha \notin AE(X,Y)$. Furthermore, one can easily see that $\alpha$ is in $AM(X,Y)$ as $\{x \in X : |x\alpha^{-1}| > 1\} = X \setminus Y$. Also, it follows directly that $\beta$ belongs to $AE(X,Y) \setminus AM(X,Y)$, as $\text{ran } \beta = Y$ and $\{x \in X : |x\beta^{-1}| > 1\} = X \setminus Z \supseteq Y \setminus Z$.  

**Theorem 2.5.** Let $S(X,Y)$ with $|Y| \geq 2$ be either the semigroup $AM(X,Y)$ or the semigroup $AE(X,Y)$. Then $S(X,Y)$ is regular if and only if $X$ is finite and $Y = X$.

Proof. The sufficiency is directly obtained from Theorem 1.2. To prove the necessity, by contrapositive, we assume that $X$ is infinite or $Y \neq X$. Then we consider it into three cases. Firstly, if $X$ is finite and $Y \neq X$, then $S(X,Y) = T(X,Y)$, and according to Theorem 1.1 it is not regular. Secondly, if $X$ is infinite and $Y \neq X$, then by Proposition 2.3 and by the assumption that $S(X,Y)$ is a semigroup, we have a bijection in $S(X,Y)$; therefore by Theorem 2.1 it is not regular. Lastly, the case that $X$ is infinite and $Y = X$ follows directly from Theorem 1.2.  

**Theorem 2.6.** Let $X$ be an infinite set. The semigroup $AM(X,Y) \cap AE(X,Y)$ is regular if and only if $Y = X$.

Proof. To prove the forward implication, by contrapositive and Proposition 2.3, we assume that $Y$ is an infinite proper subset of $X$ with $X \setminus Y$ finite. Then the semigroup $AM(X,Y) \cap AE(X,Y)$ contains all bijections from $X$ to $Y$. Now, by Theorem 2.1, we have the necessity.

For the other implication, we let $\alpha$ be an element in $AM(X) \cap AE(X)$. For each element $y$ in $\text{ran } \alpha$, we choose an element $z_y$ in $y\alpha^{-1}$. By defining $\beta : X \to X$ as in Theorem 2.1, we have $\alpha \beta \alpha = \alpha$. In addition, $\{x \in X : |y\beta^{-1}| > 1\} \subseteq (X \setminus \text{ran } \alpha) \cup \{\alpha\}$. Since $\alpha$ is an almost onto transformation, $X \setminus \text{ran } \alpha$ is finite. Then $\beta$ is in $AM(X)$. To see that $\beta$ belongs to $AE(X)$ we consider the set $X \setminus \text{ran } \beta \subseteq X \setminus \{z_x : x \in \text{ran } \alpha\}$. Since $\alpha$ is in $AM(X)$, the set $X \setminus \{z_x : x \in \text{ran } \alpha\}$ must be finite. Then $\beta$ is in $AE(X)$.  

Note that when $X$ is finite, we have $AM(X,Y) \cap AE(X,Y) = T(X,Y)$, and by Theorem 1.1, it is regular if and only if $Y = X$ or $Y$ is a singleton.

**Corollary 2.7.** $AM(X) \cap AE(X)$ is a regular semigroup.

When $X$ is infinite and $Y$ is a proper subset of $X$ with $X \setminus Y$ finite, every bijection from $X$ to $Y$ is a nonregular element in $T(X,Y)$ and in its subsemigroups, including $AM(X,Y)$, $AE(X,Y)$ and $AM(X,Y) \cap AE(X,Y)$. We next show that, apart from the bijections, there is some other kind of nonregular elements.

**Proposition 2.8.** If $X$ is an infinite set and $Y$ a proper subset of $X$ with $X \setminus Y$ finite, then there exists an element $\alpha$ in $AM(X,Y) \cap AE(X,Y)$ that is not regular in $T(X,Y)$.  

Proceedings of AMM 2017  
ALG-02-4
Proof. Let \( B \) be a nonempty finite subset of \( Y \) and choose an element \( y \) in \( B \). By assumption, \( Y \) is infinite and there exists a bijection \( \varphi \) from \( Y \) to \( Y \setminus B \). We define \( \alpha \) to be a transformation from \( X \) to \( Y \) by letting \( x\alpha = x\varphi \) if \( x \) is in \( Y \); otherwise \( x\alpha = y \). Then \( \alpha \) belongs to \( \text{AM}(X, Y) \cap \text{AE}(X, Y) \), since \( \{ x \in X : |x\alpha^{-1}| > 1 \} \subseteq X \setminus Y \) and \( \text{ran} \alpha = (Y \setminus B) \cup \{ y \} \). As \( Y\alpha \neq \text{ran} \alpha \), by Theorem 2.1, we have \( \alpha \) is not regular in \( T(X, Y) \). \( \square \)

Remark 2.9. One can see that the transformation \( \alpha \) we constructed in the above proposition is not a bijection from \( X \) to \( Y \) if \( X \setminus Y \) or \( B \) contains more than one element. Therefore, we have an infinite set of nonregular elements which are not bijections.

Now, we move into the second part of this section that concerns the regularity of \( \text{OM}(X, Y) \) and \( \text{OE}(X, Y) \). Our main result, stated in Theorem 2.13, is that when \( Y \) is a nonempty subset of \( X \), the semigroup \( \text{OM}(X, Y) \cap \text{OE}(X, Y) \) is regular if and only if \( \text{ran} \alpha = \text{ran} \) \( \alpha \), by Theorem 2.1, the proof is complete. \( \square \)

Lemma 2.10. For any infinite proper subset \( Y \) of \( X \), \( \text{OM}(X, Y) \) and \( \text{OE}(X, Y) \) have a common element that is not regular in \( T(X, Y) \).

Proof. Let \( K \) be an infinite subset of \( Y \) with \( |K| = |Y \setminus K| = |Y| \) and let \( \varphi : Y \setminus K \to K \) be a bijection. Then we fix an element \( m \) in \( Y \setminus K \) and define \( \alpha : X \to Y \) by

\[
  x\alpha = \begin{cases} 
    x\varphi & \text{if } x \in Y \setminus K, \\
    x & \text{if } x \in K, \\
    m & \text{otherwise.}
  \end{cases}
\]

Obviously, \( K \) is an infinite subset of \( \{ x \in X : |x\alpha^{-1}| > 1 \} \), and \( X \setminus \text{ran} \alpha \) actually is equal to \( X \setminus (K \cup \{ m \}) \). Hence \( \alpha \) is in the intersection of \( \text{OM}(X, Y) \) and \( \text{OE}(X, Y) \). In addition, we have \( Y\alpha = K \not\subseteq \text{ran} \alpha \). By Theorem 2.1, the proof is complete. \( \square \)

Lemma 2.11. For a finite subset \( Y \) of \( X \) with \( |Y| \geq 2 \), \( \text{OM}(X, Y) \) and \( \text{OE}(X, Y) \) have a common element that is not regular in \( T(X, Y) \).

Proof. First we choose elements \( a, b \) in \( Y \), then define \( \alpha : X \to Y \) by letting

\[
  x\alpha = \begin{cases} 
    x & \text{if } x \in Y \setminus \{ b \}, \\
    a & \text{if } x = b, \\
    b & \text{otherwise.}
  \end{cases}
\]

Then \( X \setminus \text{ran} \alpha \) is infinite, and hence \( \alpha \) is in \( \text{OE}(X, Y) \). Also, \( \alpha \) belongs to \( \text{OM}(X, Y) \) as \( (X \setminus Y) \cup \{ a, b \} = \{ x \in X : |x\alpha^{-1}| > 1 \} \) and \( X \setminus Y \) is infinite. Since \( Y\alpha \subseteq Y \setminus \{ b \} \), by Theorem 2.1, \( \alpha \) is not regular in \( T(X, Y) \). \( \square \)

As a direct consequence of Lemmas 2.10 and 2.11, we have:

Theorem 2.12. If \( Y \) is a proper subset of \( X \) with \( |Y| \geq 2 \) then \( \text{OM}(X, Y) \), \( \text{OE}(X, Y) \) and \( \text{OM}(X, Y) \cap \text{OE}(X, Y) \) are nonregular semigroups.

Theorem 2.13. \( \text{OM}(X, Y) \cap \text{OE}(X, Y) \) is regular if and only if either \( Y = X \) or \( |Y| = 1 \).

Proof. The forward implication is by Theorem 2.12. For the converse implication, if \( Y \) is a singleton, then \( \text{OM}(X, Y) \cap \text{OE}(X, Y) \) is a singleton, containing exactly one constant map; therefore it is regular. So it only remains to show that \( \text{OM}(X) \cap \text{OE}(X) \) is regular. Let \( \alpha \) be an element contained in \( \text{OM}(X) \cap \text{OE}(X) \) and let \( a \) be an element in \( X \). As before, for each \( x \) in \( \text{ran} \alpha \) we choose an element \( z_x \) in \( x\alpha^{-1} \). Then we define \( \beta : X \to X \) by letting \( x\beta = z_x \) if \( x \) is
in ran $\alpha$; otherwise $x\beta = a$. Since $\alpha$ belongs to $OE(X)$ and $X \setminus \text{ran} \alpha \subseteq \{ x \in X : |x\beta^{-1}| > 1 \}$ we have $\beta$ is in $OM(X)$.

Next we let $T = \{ z_x : x \in \text{ran} \alpha \}$. Then $X \setminus \text{ran} \beta = X \setminus (T \cup \{ a \})$ and $X \setminus T = \bigcup \{ z_x \alpha^{-1} \setminus \{ z_x \} \}$ where the union is taken over all $x$ in $\text{ran} \alpha$. Since $\alpha$ is in $OM(X)$, there must exist an element $m$ in $\text{ran} \alpha$ such that $m\alpha^{-1}$ is infinite, or $\{ z_x \in T : |z_x \alpha^{-1}| > 1 \}$ is infinite. In either case, we get that $X \setminus T$ is an infinite set, and so is $X \setminus (T \cup \{ a \})$. We have shown that $\beta$ belongs to $OE(X)$. Now, for any element $x$ in $X$, we have $x\alpha \beta \alpha = (x\alpha)\beta \alpha = z_x \alpha = x\alpha$. The proof is complete. \hfill \Box

**Corollary 2.14.** $OM(X) \cap OE(X)$ is a regular semigroup.

### 3 Supplementary Comments to $AE(X, Y)$

The reader may have an awkward feeling about the definition of $AE(X, Y)$. As defined above, $AE(X, Y) = \{ \alpha \in T(X, Y) : X \setminus \text{ran} \alpha \text{ is finite} \}$. One may wonder why it is not defined to be $\{ \alpha \in T(X, Y) : Y \setminus \text{ran} \alpha \text{ is finite} \}$, as it looks far more natural to subtract $\text{ran} \alpha$ from $Y$, instead of $X$. Let

$$\overline{AE}(X, Y) = \{ \alpha \in T(X, Y) : Y \setminus \text{ran} \alpha \text{ is finite} \}.$$  

By definition, it is clear that $AE(X, Y)$ is a subset of $\overline{AE}(X, Y)$, and they are equal if $X$ is finite. In general, an evident sufficient condition of $AE(X, Y) = \overline{AE}(X, Y)$ is that $X$ and $Y$ differ by a finite number of members; this obvious sufficient condition is also necessary. We leave the proof to the reader.

**Proposition 3.1.** $AE(X, Y) = \overline{AE}(X, Y)$ if and only if $X \setminus Y$ is finite.

Note that if $X$ is an infinite set and $Y$ is a finite subset of $X$ then $AE(X, Y)$ is the empty set and $\overline{AE}(X, Y) = T(X, Y)$.

**Theorem 3.2.** Let $X$ be an infinite set and $Y$ an infinite subset of $X$. Then $\overline{AE}(X, Y)$ is closed under composition if and only if $X \setminus Y$ is finite.

**Proof.** The sufficiency follows directly from Propositions 2.3 and 3.1. For the necessity, we prove its contrapositive. Assume that $X \setminus Y$ is infinite. We first consider the case $|X \setminus Y| < |Y|$. Let $Z$ be a subset of $Y$ with $|X \setminus Y| = |Z|$. Then there exists a bijection $\alpha$ from $X \setminus Y$ to $Z$. Also, we have a surjection $\beta$ from $Y$ to $Y \setminus Z$. Next, we define $\varphi : X \rightarrow Y$ by letting $a\varphi = a\alpha$ if $a$ belongs to $X \setminus Y$; otherwise $a\varphi = a\beta$. Then $\varphi$ is a transformation in $\overline{AE}(X, Y)$. However, $\text{ran} \varphi^2 = Y \setminus Z$, and therefore, $Y \setminus \text{ran} \varphi^2 = Z$, which is infinite. That is, $\varphi^2$ is not in $\overline{AE}(X, Y)$.

For the other case that $|X \setminus Y| \geq |Y|$ we let $\gamma$ be a surjection from $X \setminus Y$ to $Y$. Fix an element $a$ in $Y$, and define $\mu : X \rightarrow Y$ by $x\mu = x\gamma$ if $x$ belongs to $X \setminus Y$; otherwise $x\mu = a$. Then $\mu$ is a surjection and belongs to $\overline{AE}(X, Y)$. Clearly, $\text{ran} \mu^2 = \{ a \}$, so $\mu$ is in $\overline{AE}(X, Y)$, but $\mu^2$ is not. Then from both cases, $\overline{AE}(X, Y)$ is not closed. \hfill \Box

**Corollary 3.3.** Let $X$ be a set and $Y$ a nonempty subset of $X$. The following are equivalent.

(i) $X \setminus Y$ is finite,
(ii) $AE(X, Y)$ is a semigroup,
(iii) $\overline{AE}(X, Y)$ is a semigroup,
(iv) $\overline{AE}(X, Y) = AE(X, Y)$.

**Proof.** Obtained from Propositions 2.3 and 3.1 and Theorem 3.2. \hfill \Box

**References**
