Locally factorizable four-part semigroups

Kanchararat Nitkamharn\textsuperscript{1,\hspace{1pt}2} and Prakit Jampachon
Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen 40002, Thailand

Abstract

Let $S$ be a semigroup and let $E = E(S)$ be the set of all idempotents in $S$. Then $S$ is factorizable if $S = GE = EK$ for some subgroup $G$ and $K$ of $S$ and $S$ is locally factorizable if $eSe$ is factorizable for every $e \in E$. In this paper, we study locally factorizable semigroup $(O^n(\{0, 1\}); +)$ of all $n$–ary Boolean operations and we extend results to four-part semigroups.

Keywords: locally factorizable semigroups, four-part semigroups, $n$–ary Boolean operations.


1 Introduction

Four-part semigroups form a new class of semigroups which became important and interesting when sets of Boolean operations which are closed under the binary superposition operation $f + g := f(g, \ldots, g)$, were studied.

Let $O^n(\{0, 1\})$ be the set of all $n$–ary operations defined on $\{0, 1\}$, that is, $O^n(\{0, 1\}) := \{f : \{0, 1\}^n \to \{0, 1\}\}$. For $f, g \in O^n(\{0, 1\})$, we define the operation $+$ on $O^n(\{0, 1\})$ by

$$(f + g)(a_1, \ldots, a_n) := f(g(a_1, \ldots, a_n), \ldots, g(a_1, \ldots, a_n))$$

for all $(a_1, \ldots, a_n) \in \{0, 1\}^n$. Then $(O^n(\{0, 1\}); +)$ is a semigroup and called the $n$–ary Boolean operations semigroup.

In 2008, R. Butkote and K. Denecke \cite{BD2008} (see also \cite{Butkote2008}), were studied the semigroup $(O^n(\{0, 1\}); +)$ of all $n$–ary Boolean operations for $n \geq 1$. The sets $C_0^n := \{f \in O^n(\{0, 1\}) \mid f(0, \ldots, 0) = 0 \text{ and } f(1, \ldots, 1) = 1\}$, $C_1^n := \{f \in O^n(\{0, 1\}) \mid f(0, \ldots, 0) = 1 \text{ and } f(1, \ldots, 1) = 0\}$ (the notation $-C_0^n$ means that each element of this set is the negation of an element of $C_0^n$), $K_0^n := \{f \in O^n(\{0, 1\}) \mid f(0, \ldots, 0) = 0 \text{ and } f(1, \ldots, 1) = 0\}$ which contains the $n$–ary constant operation with value 0, denoted by $e_0^n$, that is, $e_0^n(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in \{0, 1\}^n$ and $K_1^n := \{f \in O^n(\{0, 1\}) \mid f(0, \ldots, 0) = 1 \text{ and } f(1, \ldots, 1) = 1\}$. $K_1^n$ contains the $n$–ary constant operation with value 1, denoted by $e_1^n$, that is, $e_1^n(a_1, \ldots, a_n) = 1$ for all $(a_1, \ldots, a_n) \in \{0, 1\}^n$. Each element of $K_1^n$ is negation of some element of $K_0^n$. Therefore,
instead of $K^n_0$ one could also write $-K^n_0$. Clearly, $O^n(\{0,1\}) = C^n_4 \cup -C^n_4 \cup K^n_0 \cup K^n_1$ is the disjoint union of these sets and in [7], show that

$$f + g = \begin{cases} 
g & \text{if } f \in C^n_4 \\
-g & \text{if } f \in -C^n_4 \\
c^n_0 & \text{if } f \in K^n_0 \\
c^n_1 & \text{if } f \in -K^n_0,
\end{cases}$$

where $-g$ is the negative of $g$, that is $-g(x) = \begin{cases} 
1 & \text{if } g(x) = 0 \\
0 & \text{if } g(x) = 1.
\end{cases}$

In this paper we study subgroups of an arbitrary four-part semigroups, determine idempotent elements, and locally factorizable semigroup ($O^n(\{0,1\}); +$) and we extend results to four-part semigroups.

2 Preliminaries

In this section, we recall some definitions, notations and properties that will be used in the later sections.

2.1 Four-part semigroups

P. Jampachon, Y. Susanti and K. Denecke [3], generalized the notion of four-part semigroups using the idea of semigroup ($O^n(\{0,1\}); +$) as follows: let $S_1 = \{a_{11}, a_{12}, \ldots, a_{1n_r}\}$, $S_2 = \{a_{21}, a_{22}, \ldots, a_{2n_s}\}$, $S_3 = \{a_{31} = a^*, a_{32}, \ldots, a_{3n_s}\}$ where $a^* \in S_3$ is a fixed element, $S_4 = \{a_4 = a^{**}, a_{42}, \ldots, a_{4n_s}\}$ where $a^{**} \in S_4$ is a fixed element, $n_r, n_s \in \mathbb{N}^*$ ($\mathbb{N}^* := \mathbb{N}\setminus\{0\}$), be four non-empty, finite and pairwise disjoint sets and let $S = S_1 \cup S_2 \cup S_3 \cup S_4$. Define a binary operation $\ast$ on $S$ by

$$a_{ij} \ast a_{lk} = \begin{cases} 
a_{lk} & \text{if } a_{ij} \in S_1 \\
a_{lk} & \text{if } a_{ij} \in S_2 \text{ where } t = \\
1 & \text{if } l = 2 \\
2 & \text{if } l = 1 \\
3 & \text{if } l = 4 \\
4 & \text{if } l = 3 \\
a^* & \text{if } a_{ij} \in S_3 \\
a^{**} & \text{if } a_{ij} \in S_4.
\end{cases}$$

It is easy to see that the binary operation $\ast$ is well defined and associative, Therefore $(S; \ast)$ is a finite semigroup. The semigroup $S = (S; \ast)$ is called a four-part semigroup.

Lemma 2.1.1. [3] Let $S$ be a four-part semigroup. Then there is a fixed point free bijective mapping $\varphi : S \longrightarrow S$ such that $\varphi \circ \varphi = id$, $\varphi(a^*) = a^{**}$, $\varphi(a^{**}) = a^*$, $\varphi(a_{1j}) = a_{2j}$, $\varphi(a_{2j}) = a_{1j}$, $\varphi(a_{3k}) = a_{4k}$ and $\varphi(a_{4k}) = a_{3k}$ for $j = 1, \ldots, n_r$ and $k = 1, \ldots, n_s$.

Note that the function $\varphi$ in Lemma 2.1.1 satisfies the following properties:

(1) $\varphi(S_1) = S_2$,
(2) $\varphi(S_2) = S_1$,
(3) $\varphi(S_3) = S_4$,
(4) $\varphi(S_4) = S_3$ and
(5) $\varphi(\varphi(a)) = a$ for all $a \in S$. 

2.2 Locally factorizable semigroups

An element \(a\) of a semigroup \(S\) is called an idempotent of \(S\) if \(a^2 = a\). Let \(E(S)\) denote the set of all idempotents of \(S\), that is, \(E(S) = \{a \in S \mid a^2 = a\}\). For \(e \in E(S)\), \(eSe\) is clearly a subsemigroup of \(S\) where \(eSe = \{exe \mid x \in S\}\), it is called a local subsemigroup.

Factorizable and locally factorizable semigroups were studied in [10], [1], [11], [12], [4], we recall some definition as follows: let \(S\) be a semigroup and let \(E = E(S)\) be the set of all idempotents in \(S\). A semigroup \(S\) is left [right] factorizable if \(S = GE[S = EK]\) for some subgroup \(G[K]\) of \(S\) ; and \(S\) is factorizable if \(S\) is both left and right factorizable. A semigroup \(S\) is locally factorizable if each local subsemigroup of \(S\) is factorizable, that is, for each \(e \in E(S)\), \(eSe = GE(eSe)[= E(eSe)K]\) for some subgroup \(G[K]\) of \(eSe\).

We will need the following result (compare Theorem 2.4 in [13] and Lemma 1 in [4 ]).

Lemma 2.2.1. [4] If \(S\) is factorizable then \(S\) has an identity and \(S = GE = EG\) where \(G\) is the group of units of \(S\). Moreover, if \(S\) has an identity and \(S = EG\) where \(G\) is the group of units of \(S\) then \(S = GE\).

In [2] (see also [5]), any semigroup \(S\), for the Green’s equivalences relation \(H\)-class of \(S\), we have any \(H\)-class of \(S\) contains at most one idempotent, any \(H\)-class of \(S\) containing an idempotent \(e\) is a subgroup of \(S\) and it is the maximum subgroup having \(e\) as its identity. Since \(H_e = eH_e \subseteq eSe\), we have that \(H_e\) is the maximum subgroup of \(eSe\) having \(e\) as its identity. But \(eSe\) is a subsemigroup of \(S\) having \(e\) as its identity, hence we have

Proposition 2.2.2. [5] Let \(S\) be a semigroup. Then:

(1) For \(e \in E(S)\), \(eSe\) is factorizable if and only if \(eSe = H_eE(eSe)\).

(2) The semigroup \(S\) is locally factorizable if and only if for each \(e \in E(S)\), \(eSe = H_eE(eSe)\).

3 Main Results

3.1 Locally Factorizable Four-part Semigroups

Proposition 3.1.1. [3] Let \(S\) be a four-part semigroup. Then \(a\) is an idempotent element of \(S\) if and only if \(a \in S_1 \cup \{a^*, a^{**}\}\).

Proof. If \(a \in S_1 \cup \{a^*, a^{**}\}\), then it is clear that \(a \ast a = a\). Conversely, let \(a \in S\) be idempotent. Suppose that \(a \notin S_1 \cup \{a^*, a^{**}\}\). If \(a \in S_2\), then \(a \ast a = \varphi(a) \neq a\), a contradiction. If \(a \in (S_3 \cup S_4) \setminus \{a^*, a^{**}\}\), then \(a \ast a \in \{a^*, a^{**}\}\) and thus \(a \ast a \neq a\), a contradiction. This completes the proof.

Theorem 3.1.2. Let \(S\) be four-part semigroup. Then a subset \(H \subseteq S\) is a subgroup of \(S\) if and only if \(H = \{a\}\) for \(a \in S_1\) or \(H = \{a, \varphi(a)\}\) for \(a \in S_2\) or \(H = \{a^*\}\) or \(H = \{a^{**}\}\) where \(a^*\) and \(a^{**}\) are fixed element in \(S_3\) and \(S_4\), respectively.

Proof. \((\Rightarrow)\) Assume that \(H\) is a subgroup of \(S\).
Case 1. \(|H| = 1\). Then \(H = \{a\}\) for some \(a \in S\) and so \(a\) is an idempotent of \(S\).
Thus \(a \in S_1 \cup \{a^*, a^{**}\}\). Hence \(H = \{a\}\) where \(a \in S_1\) or \(H = \{a^*\}\) or \(H = \{a^{**}\}\).
Case 2. \(|H| > 1\). Let \(a \in H\) such that \(a\) is not the identity of \(H\). Then \(a \notin S_1 \cup \{a^*, a^{**}\}\).
If \(a \in S_3 \cup S_4 \setminus \{a^*, a^{**}\}\), then \(a \ast a = a^* = a^* \ast a \) or \(a \ast a = a^{**} = a^{**} \ast a\).
So \(a^* = a\) or \(a^{**} = a\) a contradiction. Thus \(a \in S_2\).
This means that for all \(a \in H \setminus \{e\}\) where \(e\) is the identity of \(H\), we have \(a \in S_2\), and so \(a \ast a = \varphi(a) \in H \cap S_1\). Therefore \(\varphi(a) = e\). Suppose \(b \in H \setminus \{e\}\). Then \(b \in S_2\) and \(a \ast a = e = b \ast b\). So \(\varphi(a) = \varphi(b)\). Thus \(a = b\). Hence \(H = \{a, e\} = \{a, \varphi(a)\}\) for some \(a \in S_2\).
\((\Leftarrow)\) If \(H = \{a\}\) for \(a \in S_1\) or \(H = \{a^*\}\) or \(H = \{a^{**}\}\) where \(a^*\) and \(a^{**}\) are fixed elements in \(S_3\) and \(S_4\), respectively, then it is clear that \(H\) is a subgroup of \(S\).
If \(H = \{a, \varphi(a)\}\) for \(a \in S_2\), then we have

Proceedings of AMM 2017 ALG-01-3
and so $H$ is a subgroup.

**Corollary 3.1.3.** Let $a \in S_1 \cup S_2$. Then \( \{a, \varphi(a)\} \) is the maximum subgroup of $S$ containing $a$.

**Corollary 3.1.4.** Let $a \in \{a^*, a^{**}\}$. Then $\{a\}$ is the maximum subgroup of $S$ containing $a$.

**Theorem 3.1.5.** Let $S$ be a four-part semigroup. Then a subset $H \subseteq S$ is a maximal subgroup of $S$ if and only if $H = \{a, \varphi(a)\}$ for $a \in S_1 \cup S_2$ or $H = \{a^*\}$ or $H = \{a^{**}\}$ where $a^*$ and $a^{**}$ are fixed element in $S_3$ and $S_4$, respectively.

**Proof.** ($\Rightarrow$) Assume that $H$ is a maximal subgroup of $S$ and $H \neq \{a^*\}$ and $H \neq \{a^{**}\}$.

To show that $H = \{a, \varphi(a)\}$ where $a \in S_1 \cup S_2$. Then there exists $a \in S_1 \cup S_2$ such that $a \neq a^*$ and $a \neq a^{**}$. Since $H$ is a maximal subgroup containing $a$ and by Corollary 3.1.3, we have $H = \{a, \varphi(a)\}$.

($\Leftarrow$) It follows from Corollary 3.1.3 and Corollary 3.1.4.

**Theorem 3.1.6.** Let $S$ be a four-part semigroup with $|S_1| = |S_2| = m$ and $|S_3| = |S_4| = n$ where $m, n$ are positive integers. Then the number of all subgroups of $S$ is $2m + 2$.

**Proof.** It follows from Theorem 3.1.2.

**Lemma 3.1.7.** Let $e \in S_1$. Then $e \star S \star e = \{e, \varphi(e), a^*, a^{**}\}$.

**Proof.** Since $e \in S_1$, we have $e \star S = S$, and then

\[
e \star S \star e = S \star e = (S_1 \cup S_2 \cup S_3 \cup S_4) \star e
= \{S_1 \star e\} \cup \{S_2 \star e\} \cup \{S_3 \star e\} \cup \{S_4 \star e\}
= \{e\} \cup \{\varphi(e)\} \cup \{a^*\} \cup \{a^{**}\}
= \{e, \varphi(e), a^*, a^{**}\}.
\]

Hence $e \star S \star e = \{e, \varphi(e), a^*, a^{**}\}$.

The next lemma is easy to verify.

**Lemma 3.1.8.** Let $e \in \{a^*, a^{**}\}$. Then $e \star S \star e = \{e\}$.

**Theorem 3.1.9.** Let $S$ be a four-part semigroup. Then $S$ is factorizable if and only if $|S_1| = |S_2| = |S_3| = |S_4| = 1$.

**Proof.** ($\Rightarrow$) Assume that $S$ is factorizable. By Lemma 2.2.1, $S$ has an identity $e$. We shall show that $|S_1| = |S_2| = 1$ and $|S_3| = |S_4| = 1$. Suppose not, that is, $|S_1| = |S_2| \geq 2$ or $|S_3| = |S_4| \geq 2$.

Case 1. $|S_1| = |S_2| \geq 2$. Let $a, b \in S_1$ such that $a \neq b$. Since $e$ is an idempotent by Proposition 3.1.1 , $e \in S_1 \cup \{a^*, a^{**}\}$. If $e \in S_1$, then $a = e \star a = a \star e = e$ and $b = e \star b = b \star e = e$.

So $a = b$, a contradiction. If $e \in \{a^*, a^{**}\}$, then $a = a \star e = e \star a = e$, a contradiction to $a \in \{a^*, a^{**}\}$ and $S_1 = \emptyset$.

Case 2. $|S_3| = |S_4| \geq 2$. Let $a, b \in S_3$ such that $a \neq b$. Since $e$ is an identity, $a^* = a \star e = a$ and $a^{**} = b \star e = b$. Then $a = a^* = b$, a contradiction. Hence $|S_1| = |S_2| = |S_3| = |S_4| = 1$.

($\Leftarrow$) Assume that $|S_1| = |S_2| = |S_3| = |S_4| = 1$. Let $S_1 = \{a\}$ , $S_2 = \{b\}$ , $S_3 = \{a^*\}$ and $S_4 = \{a^{**}\}$. Then

\[
\begin{array}{c|cccc}
* & a & b & a^* & a^{**} \\
\hline
a & a & b & a^* & a^{**} \\
b & b & a & a^{**} & a^* \\
a^* & a^* & a^* & a^* & a^* \\
a^{**} & a^{**} & a^{**} & a^{**} & a^{**}
\end{array}
\]
Thus \( \{a, b\} \) is a maximal subgroup of \( S \) and \( \{a, a^*, a^{**}\} \) is a set of idempotent in \( S \).
We have \( \{a, b\} \ast \{a, a^*, a^{**}\} = \{a, b, a^*, a^{**}\} = \{a, a^*, a^{**}\} \ast \{a, b\} \). Hence \( S \) is factorizable. \( \square \)

**Theorem 3.1.10.** Let \( S \) be a four-part semigroup. Then \( S \) is locally factorizable.

**Proof.** To show that \( E(e \ast S \ast e) \ast H_e = e \ast S \ast e = H_e \ast E(e \ast S \ast e) \) for all \( e \in E(S) \), let \( e \in E(S) = S_1 \cup \{a^*, a^{**}\} \). If \( e \in S_1 \), then by Corollary 3.1.3, we have \( H_e = \{e, \varphi(e)\} \).

By Lemma 3.1.7, we get \( e \ast S \ast e = \{e, \varphi(e), a^*, a^{**}\} \). Thus \( e \ast S \ast e = \{e, \varphi(e), a^*, a^{**}\} = \{e, \varphi(e), a^*, a^{**}\} \ast \{e, \varphi(e)\} \) and \( e \ast S \ast e = \{e, \varphi(e), a^*, a^{**}\} = \{e, \varphi(e), a^*, a^{**}\} \ast \{e, \varphi(e)\} \).

Hence \( E(e \ast S \ast e) \ast H_e = e \ast S \ast e = H_e \ast E(e \ast S \ast e) \). If \( e \in \{a^*, a^{**}\} \), then by Corollary 3.1.4, we have \( H_e = \{e\} \). By Lemma 3.1.8, we get \( e \ast S \ast e = \{e\} \). Thus \( \{e\} \ast \{e\} = \{e\} \ast S \ast e = \{e\} = \{e\} \ast \{e\} \), so \( E(e \ast S \ast e) \ast H_e = e \ast S \ast e = H_e \ast E(e \ast S \ast e) \). Therefore \( S \) is locally factorizable. \( \square \)

### 3.2 Locally Factorizable Boolean Operations

In [8] R. Butkote and K. Denecke proved that \( (O^n(\{0,1\}); +) \) is a four-part semigroup where the operation + satisfies

\[
 f + g = \begin{cases} 
 g & \text{if } f \in C^n_4 \\
 -g & \text{if } f \in -C^n_4 \\
 c^n_0 & \text{if } f \in K^n_0 \\
 c^n_1 & \text{if } f \in -K^n_0 
\end{cases}
\]

for \( f, g \in O^n(\{0,1\}) \). \( -g \) is the negative of \( g \), \( c^n_0 \) is the \( n \)-ary constant operation with value 0 and \( c^n_1 \) is the \( n \)-ary constant operation with value 1.

Since \( (O^n(\{0,1\}); +) \) is a four-part semigroup, we immediately obtain the following theorems:

**Theorem 3.2.1.** Let \( O^n(\{0,1\}) \) be the set of all \( n \)-ary Boolean operations for \( n \geq 1 \). Then \( f \in O^n(\{0,1\}) \) is idempotent if and only if \( f \in C^n_4 \cup \{c^n_0, c^n_1\} \).

**Theorem 3.2.2.** Let \( O^n(\{0,1\}) \) be the set of all \( n \)-ary Boolean operations for \( n \geq 1 \). Then a subset \( H \subseteq O^n(\{0,1\}) \) is a subgroup of \( O^n(\{0,1\}) \) if and only if \( H = \{g\} \) for \( g \in C^n_4 \) or \( H = \{g, -g\} \) for \( g \in -C^n_4 \) or \( H = \{c^n_0\} \) or \( H = \{c^n_1\} \).

**Lemma 3.2.3.** Let \( g \in C^n_4 \cup -C^n_4 \). Then \( \{g, -g\} \) is the maximum subgroup of \( O^n(\{0,1\}) \) containing \( g \).

**Lemma 3.2.4.** Let \( g \in \{c^n_0, c^n_1\} \). Then \( \{g\} \) is the maximum subgroup of \( O^n(\{0,1\}) \) containing \( g \).

**Theorem 3.2.5.** Let \( O^n(\{0,1\}) \) be the set of all \( n \)-ary Boolean operations for \( n \geq 1 \). Then a subset \( H \subseteq O^n(\{0,1\}) \) is a maximal subgroup of \( O^n(\{0,1\}) \) if and only if \( H = \{g, -g\} \) for \( g \in C^n_4 \cup -C^n_4 \) or \( H = \{g\} \) for \( g \in \{c^n_0, c^n_1\} \).

**Theorem 3.2.6.** Let \( O^n(\{0,1\}) \) be the set of all \( n \)-ary Boolean operations for \( n \geq 1 \). Then the number of all subgroup of \( O^n(\{0,1\}) \) is \( 2(2^n-2) + 2 \).

**Lemma 3.2.7.** Let \( g \in C^n_4 \). Then \( g + O^n(\{0,1\}) + g = \{g, -g, c^n_0, c^n_1\} \).

**Lemma 3.2.8.** Let \( g \in \{c^n_0, c^n_1\} \). Then \( g + O^n(\{0,1\}) + g = \{g\} \).

**Theorem 3.2.9.** Let \( O^n(\{0,1\}) \) be the set of all \( n \)-ary Boolean operations for \( n \geq 1 \). Then \( O^n(\{0,1\}) \) is factorizable if and only if \( n = 1 \).

**Theorem 3.2.10.** Let \( O^n(\{0,1\}) \) be the set of all \( n \)-ary Boolean operations for \( n \geq 1 \). Then \( O^n(\{0,1\}) \) is locally factorizable.
References


